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BABYLONIAN MATHEMATICS
with Special Reference to Recent Discoveries*

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In a vice-presidential address before Section A of the American Association for the Advancement of Science just six years ago, I made a somewhat detailed survey[1] of our knowledge of Egyptian and Babylonian Mathematics before the Greeks. This survey set forth considerable material not then found in any general history of mathematics. During the six years since that time announcements of new discoveries in connection with Egyptian mathematics have been comparatively insignificant, and all known documents have probably been more or less definitively studied and interpreted. But the case of Babylonian mathematics is entirely different; most extraordinary discoveries have been made concerning their knowledge and use of algebra four thousand years ago. So far as anything in print is concerned, nothing of the kind was suspected even as late as 1928. Most of these recent discoveries have been due to the brilliant and able young Austrian scholar Otto Neugebauer who now at the age of 36 has a truly remarkable record of achievement during the past decade. It was only in 1926 that he received his doctor's degree in mathematics at Göttingen, for an interesting piece of research in Egyptian mathematics; but very soon he had taken up the study of Babylonian cuneiform writing. He acquired a mastery of book and periodical literature of the past fifty years, dealing with Sumerian, Akkadian, Babylonian, and Assyrian grammar, literature, metrology, and inscriptions; he discovered mathematical terminology, and translations the accuracy of which he thoroughly proved. He scoured museums of Europe and America for all possible mathematical texts, and translated and interpreted them. By 1929 he had founded periodicals called *Quellen und Studien zur Geschichte der Mathematik*[2] and from the first, the latter contained remarkable new information concerning Babylonian mathematics. A trip to Russia resulted in securing for the *Quellen* section, Struve's edition of the first complete publication of the Golenishchev mathematical papyrus of about 1850 B.C. The third and latest volume of the *Quellen*, appearing only about three months ago,

[1] "Mathematics before the Greeks," *Science*, n.s. v. 71, 31 Jan. 1930, p. 109–121.

[2] I shall later refer to the two periodicals simply by the words *Quellen*, and *Studien*.

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is a monumental work by Neugebauer himself, the first part containing over five hundred pages of text, and the second part in large quarto format, with over 60 pages of text and about 70 plates. This work was designed to discuss most known texts in mathematics and mathematical astronomy in cuneiform writing. And thus we find that by far the largest number of such tablets is in the Museum of Antiquities at Istanbul, that the State Museum in Berlin made the next larger contribution, Yale University next, then the British Museum, and the University of Jena, followed by the University of Pennsylvania, where Hilprecht, some thirty years ago, published a work containing some mathematical tables. In the Museum of the Louvre are 16 tablets; and then there are less than 8 in each of the following: the Strasbourg University and Library, the Musée Royaux du Cinquantenaire in Brussels, the J. Pierpont Morgan Library Collection (temporarily deposited at Yale) the Royal Ontario Museum of Archaeology at Toronto, the Ashmolean Museum at Oxford, and the Böhl collection at Leyden. Most of the tablets thus referred to date from the period 2000 to 1200 B.C. It is a satisfaction to us to know that the composition of this wonderful reference work was in part made possible by The Rockefeller Foundation. Some two years ago it cooperated in enabling Neugebauer to transfer his work to the Mathematical Institute of the University of Copenhagen, after Nazi intolerance had rendered it impossible to preserve his self respect while pursuing the intellectual life. This new position offered the opportunity for lecturing on the History of Ancient Mathematical Science. The first volume of these lectures,[3] on "Mathematics before the Greeks," was published last year, and in it are many references to results, the exact setting of which are only found in his great source work referred to a moment ago. In these two works, then, we find not only a summing up of Neugebauer's wholly original work, but also a critical summary of the work of other scholars such as Frank, Gadd, Genouillac, Hilprecht, Lenormant, Rawlinson, Thureau-Dangin, Weidner, Zimmern, and many others.[4] Hence my selection of material to be presented to you to-night will be mainly from these two works. Before turning to this it may not be wholly inappropriate to interpolate one remark regarding Neugebauer's service to mathematics in general. Since 1931 his notable organizing ability has been partially occupied in editing and directing two other periodicals, (1) *Zentralblatt für Mathematik* (of which 11 volumes have already appeared), and (2) *Zentralblatt für Mechanik*, (3 volumes)—a job which of itself would keep many a person fully employed. *Mais, revenons à nos moutons!*

From about 3500 to 2500 years before Christ, in the country north of the Persian Gulf between the Tigris and Euphrates Rivers, the non-semitic Sumerians, south of the semitic Akkadians, were generally predominant in Babylonia. By 2000 B.C. they were absorbed in a larger political group. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of the rivers regulated by canals, went back to Sumerian times, like also a considerable part of their religion and law, and their system of mathematics, except, possibly, for certain details. As to mathematical transactions we find that long before coins were in use the custom of paying interest for the loan of produce, or of a certain weight of a precious metal,

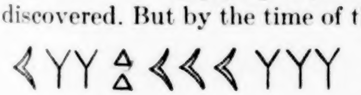
[3] *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, v. 1, *Vorgriechische Mathematik* (Die Grundlehren der mathematischen Wissenschaften, v. 43), Berlin, 1934. Reference to this work will later be made simply by the word *Vorlesungen*.

[4] For the literature of Babylonian mathematics prior to 1929, see my *Bibliography* in the Chace-Manning-Bull edition of the *Rhind Mathematical Papyrus*, v. 2; for later items see K. Vogel's bibliography in *Bayer. Blätter f. d. Gymnasialschulwesen*, v. 71, 1935, p. 16–29.

was common. Sumerian tablets indicate that the rate of interest varied from 20 per cent to 30 per cent, the higher rate being charged for produce. At a later period the rate was $5\frac{1}{2}$ per cent to 25 per cent for metal and 20 per cent to $33\frac{1}{3}$ per cent for produce.[5] An extraordinary number of tablets show that the Sumerian merchant of 2500 B.C. was familiar with such things as weights and measures, bills, receipts, notes and accounts.

Sumerian mathematics was essentially sexagesimal and while a special symbol for 10 was constantly used it occupied a subordinate position; there were no special symbols for 100 or for 1000. One hundred was thought of as $60+40$ and 1000 as $16\cdot 60+40$. But in these cases the Sumerian would write simply 1,40 and 16,40;

$=12\times 60^2+25\times 60+33=44733$. Hence the Sumerians had a relative positional notation for the numbers. The word cuneiform means wedgeshaped and the numbers from one to nine were denoted by the corresponding number of wedges, where the Egyptian simply employed strokes. For 10, as we have seen, an angle-shaped sign was used. Practically all other integers were made up of combinations of these in various ways. There is great ambiguity because, for example, a single upright wedge may stand for 1 or 60 or any positive or negative integral multiple of 60. **Hence there was a special sign D for 60; D or D or K for 600, the last of which suggests 60×10; O for 3600; and ◎ for** 36000, again suggesting a product. No special sign for zero in Sumerian times, other than an empty space, has yet been discovered. But by the time of the Greeks



$=12\times 60^2$ **$+0+33=43233$, the sign ≙ being for** zero. But to matters of numeral notation we shall make no further reference, except to remark that the Babylonians thought of any positive integer $a=\sum c_n 60^n$, and in the form

$$a=\cdots c_2c_1c_0c_{-1}c_{-2}\cdots.$$

This may not, of course, correspond to what we call integers. By means of negative values of n, fractions were introduced.

Babylonian multiplication tables are very numerous and are often the products of a certain number, successively, by 1, 2, 3 $\cdots$ 20, then 30, 40 and 50. For example, on tablets of about 1500 B.C. at Brussels are tables of 7, 10, $12\frac{1}{2}$, 16, 24, each multiplied into such a series of numbers. There are various tablets giving the squares of numbers from 1 to 50, and also the cubes, square roots and cube roots of numbers. But we must be careful not to assume too much from this statement; the tables of square roots and cube roots were really exactly the same as tables of squares and cubes, but differently expressed. In the period we are considering the Egyptian really had nothing to correspond to any of these tables, nor do we know that even the conception of cube root was within his ken. Until two years ago it was a complete mystery why the Babylonians had tables of cubes and cube roots, but finally a tablet in the Berlin Museum gave a clue. This is a table of n^3+n^2, for $n=1$ to 30. Certain problems on British Museum tablets were found to lead to cubic equations of the form $(ux)^3+(ux)^2=252$. Hence Neugebauer reasoned in his article of 1933 in the Göttingen *Nachrichten* that the purpose of the tablet in question was to solve cubic equations in this "normal form." He contended that it was within the power of the Babylonians, by a linear transformation $z=x+c$, to reduce a four-term cubic equation $x^3+a_1x^2+a_2x+a_3=0$ to $z^3+b_1z^2+b_2=0$. Multiplying this equation by $1/b_1^3$ we have at once (on setting $z=b_1w$, and $a=-b_2/b_1^3$) the normal form

[5] M. Jastrow, Jr., *The Civilization of Babylonia and Assyria*, Philadelphia, 1915, p. 326, 338; C. H. W. Johns, *Babylonian and Assyrian Laws, Contracts and Letters*, New York, 1904, p. 251, 255–256. See also D. E. Smith, *History of Mathematics*, vol. 2, 1925, p. 560.

$$w^3+w^2=a.$$

Neugebauer's theory as to the possibility of such a reduction is in part based on problems to which I shall later refer. Up to the present, however, Neugebauer has found no four-term cubic equation solved in this way. And indeed in these same British Museum tablets are two problems which lead naturally to such equations but are solved by a different method.[6]

Neugebauer feels that *tables are the foundation of all discussion of Babylonian mathematics*, that more tables, such as the one to which we have just referred, are likely to be discovered, and to illuminate other mathematical operations. There are many tables of parallel columns of integers such as

2	30
3	20
4	15
5	12
6	10
8	7, 30
9	6, 40

which is nothing but a table of reciprocals $1/n=\bar{n}$ in the sexagesimal system. $n\cdot\bar{n}$ is always equal to 60 raised to 0, or some positive or negative integral power. It is notable that in the succession of numbers chosen, the divisor $n=7$ does not appear, the reason being that there is no integer $\bar{n}$ such that the product is the power of 60 indicated. Hence every divisor, a, with a corresponding $\bar{n}$ must be of the form $a=2^\alpha\cdot 3^\beta\cdot 5^\gamma$. All such reciprocals are called regular; and such reciprocals as of 7 and 11 irregular.[7] When irregular numbers appear in tables the statement is made that they do not divide.

Some of these tables are extraordinary in their complexity and extent. One tablet in the Louvre, dating from about the time of Archimedes, has nearly 250 reciprocals of numbers many of them six-place, and some seven. For example, here is the second last entry for a six-place number:[8] 2, 59, 21, 40, 48, 54   20, 4, 16, 22, 28, 44, 14, 57, 40, 4, 56, 17, 46, 40 that is, the product of $(2\times 60^5+59\times 60^4+\cdots+54)\times(20\times 60^{13}+4\times 60^{12}+\cdots+40)=60^{19}$.

The object of a table of reciprocals is to reduce division to multiplication since b/a equals b multiplied by the reciprocal of a.

I referred a few moments ago to one-figure tables of squares (that is, the squares of numbers from 1 to 60). In a tablet of the Ashmolean Museum at Oxford is the only example at present known of a two-figure table of squares.[9] This dates from about 500 B.C. The tablet is of further interest from the fact that on it are several examples of the sign for zero, e.g.

$$(15,\ 30)^2=\ 4,\ \cdot\ ,\ 15$$

$$(39,\ 30)^2=26,\ \cdot\ ,\ 15.$$

The latter is equivalent to $2370^2=5{,}616{,}900$

Among table-texts are also certain ones involving exponentials. From Neugebauer's volume of Lectures we may easily gain the impression[10] that these are tables for c^n, $n=1$ to 10, for $c=9$, $c=16$, $c=100$, and $c=225$. On turning, however, to his work published three months ago we find that the tables in question are on Istanbul tablets, which are in very bad condition, so that for $c=16$ there is not a single complete result; for $c=9$ there are only three complete results, and similarly for the others. Enough is present however to show that the original was probably at one time as described.

One use of such tablets is in solving problems of compound interest. For example in a Louvre text dating back to

[6] Compare Göttingen, *Nachrichten, Math.-phys. Kl.*, 1933, p. 319; also *K. Danske Videnskabernes Selskab, Mathem.-fysiske Meddelelser*, v. 12, no. 13, p. 9. Also *Quellen*, v. 3, part 1, p. 200–201, 210–211.

[7] It is easy to approximate to 1/7, e.g. 7/28 = ; 8, 45; 13/90 = ; 8, 40, etc., but there is no case known where this was done.

[8] *Quellen*, v. 3, part 1, p. 22.

[9] *Quellen*, v. 3, part 1, p. 72–73 and part 2, plate 34.

[10] *Vorlesungen*, p. 201; *Quellen*, v. 3, part 1, p. 77–79 and part 2, plate 42.

2000 B. C. is a question as to how long it would take for a certain sum of money to double itself at 20% interest.[11] The problem here, then, is to find x in the equation

$$(1;\,12)^x = 2.$$

The answer given is $4-0;\ 2,\ 33,\ 20 = 3;\ 57,\ 26,\ 40$ years, not so very different from the more accurate result 3; 48. That is, from $(1;\,12)^4 = 2;\ 4,\ 24,\ 57,\ 36\ 4$ was found too large, giving a quantity greater than 2. How the amount to subtract was discovered is not indicated in the text, and can not now be surmised. This is a conspicuous example of a solution by the Babylonians of an equation of the type $a^x = b$ where x was not integral.

Both in the Berlin Museum and in Yale University are tablets with other problems in compound interest. If for no other reason than to point out that five-year plans are not wholly a modern invention I may refer to a problem in a Berlin papyrus, the transcription and discussion of which occupies 16 pages of Neugebauer's new book.[12] As yet I have not mastered all the discussion of this problem, but certain facts can be stated with assurance. There is a very curious combination of simple and compound interest which is naturally suggestive of what may have been customary in old Babylonia. If P is an amount of principal, r is the rate of interest per year (here 20%), and we suppose that through a five-year period P accumulates at simple interest it will amount to $2P$ at the end of the first five-year period. This amount $2P$ is then put at interest in the same way for a second five-year period and the principal is again doubled to 2^2P. The amount of capital at the end of any year is therefore given by the formula

$$A = 2^nP(1+rm)$$

where $0 \leqq m < 5$, and n is the number of five-year periods. One of the problems is: How many five-year periods will it take for a given principal P to become a given sum A? The particular case when $m=0$ gives us the equation $A = 2^nP$. In modern notation $n = \log_2 A/P$. Now Neugebauer suggests as a possible theory in explanation of the text that something equivalent to logarithms to the base 2 was here used. In the problem $P=1$, $A=1,4$ whence $n=6$.

Two other suggestive problems of the Babylonians, involving powers of numbers are in a Louvre tablet of about the time of Archimedes.[13] We have here 10 terms of a geometric series in which the first term is 1, and 2 the constant multiplier; the sum is given correctly,

$$\sum_{i=0}^{9} 2^i = 1+2+2^2+\ \cdots\ +2^9 = 1023.$$

But what is of special interest is the apparent suggestion as to how this number 1023 was obtained. On the tablet it is stated that it is the sum of $511 = 2^9-1$, and $512 = 2^9$. That is

$$1023 = 2^9+2^9-1 = 2\cdot 2^9-1 = 2^{10}-1.$$

Does this imply a knowledge of Euclid's formula leading to the sum of the ten terms of the geometric progression as, $(2^{10}-1)/(2-1)$?

On the same tablet is the following

$$1\cdot 1+2\cdot 2+\ \cdots\ +10\cdot 10 = (1\cdot 1/3+10\cdot 2/3)\cdot 55 = 385.$$

That is, we have the sum of the squares of the first 10 integers, and this sum is the product of two integers, one of which is 55, the sum of the first 10 integers. In general terms this relation may be stated

$$\sum_{i=1}^{n} i^2 = (1\cdot 1/3+n\cdot 2/3)\sum_{i=1}^{n} i.$$

Now if we set $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, a formula known to the Pythagoreans, we have $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$. This formula is practically equivalent to one known to Archimedes.

[11] *Quellen*, v. 3, part 2, p. 37–38, 40–41.

[12] *Quellen*, v. 3, part 1, p. 351–367, and part 2, plates 29, 32, 54, 56, 57; *Vorlesungen*, p. 197–199.

[13] *Quellen*, v. 3; part 1, p. 96–97, 102–103 and part 2, plate 1; *Studien*, v. 2, 1932, p. 302–303.

Turning back to tables for a moment one finds a word for subtraction, *lal*; 19 is 20 *lal* 1, 37 is 40 *lal* 3; a *lal* $b=a-b$. Neugebauer refers to a late astronomical text in which *before* each of 12 numbers the words *tab* and *lal* (plus and minus)[14] are placed, suggesting the arrangements of points above and below a line which lie on a wave-shaped curve. This seems extraordinary. Neugebauer promised more about the matter in the third volume of his Lectures which is to deal with mathematical astronomy.

It is also a matter of great historical interest that, in at least three different problems in simultaneous equations in two unknowns, *negative numbers occur as members*. These examples are in Yale University texts,[15] and it is very noteworthy that such conceptions were not current in Europe, even 2500 years later.

As a point of departure for certain other things let us now consider some geometrical results known to the Babylonians. There will of course be no misunderstanding when I state general results. These simply indicate operations used in many *numerical* problems of the Babylonians.

1. The area of a *rectangle* is the product of the lengths of two adjacent sides.

2. The area of a *right triangle* is equal to one half the product of the lengths of the sides about the right angle.

3. The sides about corresponding angles of *two similar right triangles* are proportional.

4. The area of a *trapezoid* with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.

5. The perpendicular from the vertex of an *isosceles triangle* on the base, bisects the base. The area of the triangle is the product of the lengths of the altitude and half the base.[16] Indeed the Babylonians would probably think of the area of a triangle, other than right or isosceles, as the product of the lengths of its base and altitude—an easy deduction from two adjacent, or overlapping, right triangles. A large rectilineal area portrayed in a Tello tablet, in the Museum at Istanbul,[17] was calculated by dividing it up into 15 parts: 7 right triangles, 4 rectangles (approximately), and 4 trapezoids.

6. The *angle in a semicircle* is a right angle, a result till recently first attributed to Thales of Miletus, who flourished 1500 years later.

7. $\pi=3$, and the *area of a circle* equals one twelfth of the square of the length of its circumference (which is correct if $\pi=3$). $A=\pi r^2=(2\pi r)^2/4\pi$.

8. The *Pythagorean theorem*, a result entirely unknown to the Egyptian.[18]

9. The volume of a *rectangular parallelopiped* is the product of the lengths of its three dimensions, and the volume of a *right prism* with a trapezoidal base is equal to the area of the base times the altitude of the prism. Such a volume as the latter would be considered in estimating the amount of earth dug in a section of a canal. In a British Museum tablet the volume of a solid equivalent to that cut

[14] *Vorlesungen*, p. 18.

[15] *Quellen*, v. 3, part 1, p. 387, 440, 447, 455, 456, 463, 470, 474; and part 2, plates 23, 48, 59.

[16] *Quellen*, v. 3, part 2, p. 43, 46–47, 50–51, and part 1, p. 97, 104.

[17] A. Eisenlohr, *Ein altbabylonischer Feldplan*, Leipzig, 1896. J. Oppert, Académie d. Inscriptions et Belles-Lettres, *Comptes Rendus* s. 4, v. 24, 1896, p. 331–348; also in *Revue d'Assyriologie et d'Archéologie Orientale*, v. 4, 1897, p. 28–33. F. Thureau-Dangin, *Revue d'Assyriologie*, v. 4, 1897, p. 13–27.

[18] After many mistatements by mathematical historians it was an Egyptologist, the late T. E. Peet, in his *Rhind Mathematical Papyrus* (London, 1923, p. 31–32) who brought out the fact that there is not one scrap of evidence that the Egyptians knew the Pythagorean theorem, even in the simple 3–4–5 case. He gave also interesting new information about the *harpedonaptai*, or rope stretchers, referred to by Democritus. It is of course true that there are problems involving the relations of such numbers as 8, 6, and 10, as in Berlin Papyrus 6619 (about 1850 B. C.): Distribute 100 square ells between two squares whose sides are in the ratio 1 to $\frac{3}{4}$. The same equations arise in problem 6 of the Golenishev papyrus: Given that the area of a rectangle is 12 arurae and the ratio of the lengths of the sides $1:\frac{1}{2}\ \frac{1}{4}$, find the sides; see also the Kahun papri, ed. by Griffith (1898).

off from a rectangular parallelopiped by a plane through a pair of opposite edges is given correctly as half that of the parallelopiped.[19]

10. The volume of a *right circular cylinder* is the area of its base times its altitude.

11. The volume of the *frustrum of a cone* is equal to its altitude multiplied by the area of its median cross-section.[20]

12. The volume of the *frustrum of a cone*, or of a square pyramid, is equal to one-half its altitude multiplied by the sums of the areas of its bases. [Contrast this approximation to the volume of a frustrum of a square pyramid with the exact formula known to the Egyptians of 1850 B. C., $V=\frac{1}{3}h(a_1^2+a_1a_2+a_2^2)$, where a_1, a_2 are the lengths of sides of the square bases, and h the distance between them.] On the other hand Neugebauer believes that the Babylonians also had an exact value for the volume of the frustrum of a square pyramid, namely[21]

$$V=h\left[\left(\frac{a_1+a_2}{2}\right)^2+\frac{1}{3}\left(\frac{a_1-a_2}{2}\right)^2\right];$$

concerning the second term there has been more than one discussion.

Practically all of these results are in British Museum texts of 2000 B. C.

That the Pythagorean theorem was known to the Babylonians of 2000 B. C. is certain from the following problems of a British Museum text:[22] (1) To calculate the length of a chord of a circle from its *sagitta* and the circumference of the circle; and (2) To calculate the length of the *sagitta* from the chord of a circle, and its circumference. If c be the length of the chord, a of its *sagitta* and d of the diameter (one-third of the circumference) of the circle, the formulae used are evidently

$$c=\sqrt{[d^2-(d-2a)^2]}$$

$$a=\tfrac{1}{2}[d-\sqrt{(d^2-c^2)}].$$

Now every step of the numerical work is equivalent to substitution in these formulae.

The same is true of the following problem in another British Museum tablet.[23] A beam of given length l was originally upright against a vertical wall but the upper end has slipped down a given distance h, what is the distance d of the other end from the wall? Each step is equivalent to substitution in the formula

$$d=\sqrt{[l^2-(l-h)^2]}$$

and then follows the converse problem, given l and d to find h,

$$h=l-\sqrt{l^2-d^2}.$$

In these problems a, c, d, h, and l are all integers.

A third problem involving the use of the Pythagorean theorem is one on a Louvre tablet of the Alexandrine period:[24] Given in a rectangle that the sum of two adjacent sides and the diagonal is 40 and that the product of the sides is 120. The sides are found to be 15 and 8 and the diagonal 17.

There are, however, various problems in Babylonian mathematics where square roots of non-square numbers, such as 1700, are discussed. In this particular case the problem, on an Akkadian tablet of about 2000 B.C., is to find the length of the diagonal of a rectangle whose sides are ;40 and ;10. It is worked out twice, as if by two approximation formulae.[25] If the lengths of the diagonal and sides of a rectangle are respectively d, a, and b, $d=\sqrt{(a^2+b^2)}$ and the approximation formulae are:

[19] *Quellen*, v. 3, part 2, p. 43, 47, 52.

[20] *Studien*, v. 1, 1929, p. 86–87; *Vorlesungen*, p. 171; *Quellen*, v. 3, part 1, p. 176.

[21] *Studien*, v. 2, 1933, p. 348–350; *Vorlesungen*, p. 171; *Quellen*, v. 3, part 1, p. 150, 162, 187–188. Heron of Alexandria (second century A. D.?) found the volume of such a pyramid, for which $a_1=10$, $a_2=2$, $h=7$ (*Heronis Alexandrini Opera quae supersunt omnia*, Leipzig, v. 5, 1914, p. 30–35), every step being equivalent to substituting in this formula.

[22] *Studien*, v. 1, 1929, p. 90–92.

[23] *Quellen*, v. 3, part 2, p. 53.

[24] *Studien*, v. 2, 1932, p. 294; *Quellen*, v. 3, part 1, p. 104.

[25] *Studien*, v. 2, 1932, p. 291–294; *Vorlesungen*, p. 33–36; *Quellen*, v. 3, part 1, p. 279–280, 282, 286–287, and part 2, plates 17, 44.

(1) $$d=a+\frac{b^2}{2a};$$

(2) $$d=a+2ab^2.$$

The first of these is equivalent to the one employed several times, two thousand years later, by Heron of Alexandria, in his *Metrica*.

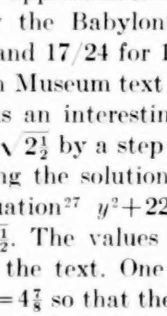
On the other hand the dimensions of the second formula are incorrect. After considerable calculation Neugebauer shows that a correct and good approximation to d is given by the following

(3) $$d=a+\frac{2ab^2}{2a^2+b^2}.$$

Since, in the particular problem in question, $1/(2a^2+b^2)=12/11$ that is, almost unity, Neugebauer has surmised that the equivalent of this third formula may have been used.

The Heron approximation formula was also used by the Babylonians to find[26] $1\frac{5}{12}$ for $\sqrt{2}$ and 17/24 for $1/\sqrt{2}$.

In a British Museum text of about 2000 B. C. there is an interesting attempt to approximate $\sqrt{2\frac{1}{2}}$ by a step equivalent to that of seeking the solutions of the Diophantine equation[27] $y^2+22\frac{1}{2}=x^2$. When $x=5$, $y=\sqrt{2\frac{1}{2}}$. The values $y=1\frac{1}{2}$, $x=5\frac{1}{4}$ are found in the text. One readily finds also $y=1\frac{1}{8}$, $x=4\frac{7}{8}$ so that the required y is between $1\frac{1}{8}$ and $1\frac{1}{2}$. Babylonian tables of squares might well give much closer approximations.



Two of the geometrical theorems referred to, a few moments ago, are employed in the solution of the following problem of a Strasbourg tablet:[28] Consider two adjacent trapezoids, sections of the same right triangle and with a common side of length c as in the figure. The upper area of height h_u (between u and c) is given as 783; the lower area of height h_l (between c and l) is 1377. It is further given that

(1) $$h_l=3h_u$$

(2) $$u-c=36$$

Then by applying the theorems mentioned

(3) $$h_u\cdot\frac{u+c}{2}=783$$

(4) $$h_l\cdot\frac{c+l}{2}=1377$$

(5) $$u-c=(1/3)(c-l)$$

five equations from which the five unknown quantities are found.

There are many similar problems, one, of a group, leading to ten equations in ten unknowns. This is in connection with the division of a right triangle (by lines parallel to a side) into six areas of equal altitudes, while their areas are in arithmetic progression.[29] This problem seems to show mathematics studied for its own sake, just as problem 40 of the Rhind papyrus suggested a similar thought there.

Consider now another Strasbourg problem, of a different type, leading to a quadratic equation:[30] The sum of the areas of two squares is a given area. The length (y) of the side of one square exceeds a given ratio (α/β) of the length (x) of the side of the other square, by a quantity d. The problem is to find x and y. Here

$$x^2+y^2=A,$$

$$y=\frac{\alpha}{\beta}x-d.$$

[26] *Studien*, v. 2, 1932, p. 294–295; *Vorlesungen*, p. 37; *Quellen*, v. 3, part 1, p. 100, 104, and part 2, plate 1.

[27] *Studien*, v. 2, 1932, p. 295–297, 309; *Quellen*, v. 3, part 1, p. 172.

[28] *Studien*, v. 1, 1929, p. 67–74; *Quellen*, v. 3, part 1, p. 259–263.

[29] *Quellen*, v. 3, part 1, p. 253; *Studien*, v. 1, 1929, p. 75–78; *Vorlesungen*, p. 180–181.

[30] *Studien*, v. 1, 1930, p. 124–126; *Quellen*, v. 3, part 1, p. 246–248.

If we set $x = X\beta$ it may be readily shown that we are led to the equation

$$X^2 - \frac{2d\alpha}{\alpha^2+\beta^2}X - \frac{A-d^2}{\alpha^2+\beta^2} = 0$$

whence

$$X = \frac{1}{\alpha^2+\beta^2}\{d\alpha + \sqrt{d^2\alpha^2 + (\alpha^2+\beta^2)(A-d^2)}\}$$

Now every step of the solution of this problem is equivalent to substitution in this formula.

There are scores of problems which prove this amazing fact, that the Babylonians of 2000 B. C. were familiar with our formula for the solution of a quadratic equation. Until 1929 no one suspected that such a result was known before the time of Heron of Alexandria, two thousand years later.

In general only the positive sign before the radical in the solution of a quadratic equation is to be considered; but in the following problem[31] (because of its nature) both roots are called for. The problem on a Berlin tablet deals with the dimensions of a brick structure of given height h, of length l of width w, and of given volume v. The exact nature of the structure is not clear but it is given that $v/a = hlm$, where $1/a$ is a given numerical factor. $l+m$ is also a given quantity S; it is required to find l and m. They are evidently roots of the quadratic equation

$$X^2 - SX + \frac{v}{ah} = 0;$$

when l and m are given by

$$\frac{S}{2} \pm \sqrt{\left(\left(\frac{S}{2}\right)^2 - \frac{v}{ah}\right)}.$$

The upper sign gives the required value for l and the lower for m. Of course both roots are positive.

On another Berlin tablet[32] is a problem divorced from geometrical connections but which may possibly illustrate another point of interest. Two unknowns y_1, y_2 are connected by relations

$$(1) \qquad y_1 - \frac{\alpha}{\beta}(y_1+y_2) = D$$

$$(2) \qquad y_1y_2 = 1$$

where α, β, D are given, $\beta > \alpha$. New variables are then introduced,

$$(3) \qquad x_1 = (\beta-\alpha)y_1,\ x_2 = \alpha y_2$$

whence $x_1 - x_2 = \beta D$, $x_1x_2 = \alpha(\beta-\alpha)$. From the resulting quadratic equation

$$X^2 - \beta DX - \alpha(\beta-\alpha) = 0$$

x_1 and $-x_2$ are found to be,

$$\pm\frac{\beta D}{2} + \sqrt{\left[\left(\frac{\beta D}{2}\right)^2 + \alpha(\beta-\alpha)\right]},$$

y_1 and y_2 are then found from (3). Neugebauer emphasizes that here, and in other texts we have a transformation of a quadratic equation to a *normal form* with unity as coefficient of the squared term. And also we have another example of an equation in which both roots are positive and the double sign before the radical is taken in solving the question.

We have now considered Babylonian solutions of simultaneous equations, exponential equations, quadratic equations, and cubic equations. Before giving examples leading to equations of higher degree some general remarks may be made about 17 of the 35 mathematical tablets at Yale.[33] In size they are from 9.5×6.5 cm. to 11.5×8.5 cm. They belong to series and contain the enunciation of problems systematically arranged. No solutions are given. On one tablet there are 200 problems and on the seventeen over 900. Since only a few tablets have been preserved there must have been thousands of problems in the original series.

To give an idea of what is meant by problems being arranged in a series it may

[31] *Quellen*, v. 3, part 1, p. 280–281, 283–285.

[32] *Quellen*, v. 3, part 1, p. 350–351; *Vorlesungen*, p. 186–187.

[33] *Quellen*, v. 3, part 1, p. 381–516, and part 2, plates 36, 37, 57–59, p. 60–64; *Studien*, v. 3, 1934, p. 1–10.

be noted that on one tablet are 55 problems of the type to find x and y, given[34]

$$\begin{cases} xy = 600 \\ (ax+by)^2+cx^2+dy^2=B \end{cases}$$

where some coefficients can be zero. The first equation is the same for all of these problems. The second equations for the first seven problems are as follows:

$$\begin{array}{ll} 1. & (3x)^2+\ y^2=\ 8500 \\ 2. & \qquad\quad +2y^2=\ 8900 \\ 3. & \qquad\quad -\ y^2=\ 7700 \\ \hline 4. & (3x+2y)^2+\ x^2=17800 \\ 5. & \qquad\quad +2x^2=18700 \\ 6. & \qquad\quad -\ x^2=16000 \\ 7. & \qquad\quad -2x^2=15100 \end{array}$$

Problems 48 and 49 are

$$[3x+5y-2(x-y)]^2-2y^2=28100$$
$$+x^2+y^2=30200.$$

The solution of all the equations leads at once to a biquadratic equation which is a quadratic equation in x^2.

On another tablet, however, are problems of the type[35]

$$xy=A$$
$$a(x+y)^2+b(x-y)+C=0$$

which leads to the most general form of biquadratic equation (if $d=c+2aA$)

$$x^4+\frac{b}{a}x^3+\frac{d}{a}x^2-\frac{bA}{a}x+A^2=0.$$

The second equation of one of the problems in this group is

$$\tfrac{1}{5}(x+y)^2-60(x-y)=-100,$$

one of the extraordinary examples of a negative number in the right hand member, to which we have already referred.

Problems on another tablet[36] lead to the most general cubic equation. How the Babylonians found the solution of such equations is unknown. It is true that $x=30$, $y=20$ gives the solution of every one of these, and of hundreds of problems in other series; Neugebauer believes, however, that it is nonsensical to imagine that such values were merely to be guessed (*Quellen*, v. 3, part 1, p. 456).

There are problems about measurement of corn and grain, workers digging a canal, interest for loan of silver, and more problems like the Strasbourg texts where algebraic questions are derived from consideration of sections of a triangle. Neugebauer concludes the second part of his great work with an italicized statement to the effect that the Strasbourg and Yale texts prove that the chief importance of Babylonian mathematics lies in algebraic relations—not geometric.

In his work of thirty years[37] ago Hilprecht was guilty of more than one disservice to truth. One such was his great emphasis on mysticism in Babylonian mathematics, its association with what he called "Plato's number," $60^4=12{,}960{,}000$. In spite of the protests of contemporary scholars such ideas were widely disseminated. We have noted enough to realize that such an idea is purest bunk—rather freely to translate Neugebauer's expression.[38]

While it has been possible for me to draw your attention to only a few somewhat isolated facts, I trust that you have received the impression that in Babylonian algebra of 4000 years ago we have something wonderful, *real* algebra without any algebraic notation or any actual setting forth of general theory. And if all this was known in 2000 B. C., how far back must we go for the beginnings of the Sumerian mathematics, simple arithmetic operations? Probably back to 3000 B. C. at least.

[34] *Quellen*, v. 3, part 1, p. 418–420.
[35] *Quellen*, v. 3, part 1, p. 455–456.
[36] *Quellen*, v. 3, part 1, p. 402.

[37] *Mathematical, Metrological and Chronological Tablets, from the Temple of Nippur* (Univ. Pennsylvania), s.A, Cuneiform Texts, v. 20, part 1, Philadelphia, 1906.
[38] *K. Danske Videnskabernes Selskab, Matem.-fysiske Meddelelser*, v. 12, no. 13, 1934, p. 5–6.

One thing which is of great interest in the study of Egyptian and Babylonian mathematics is, that we handle and study the actual documents which go back to those days, four thousand years ago. Contrast with this the way in which we learn of Greek mathematics. Even in the case of such a widely used work as Euclid's Elements, there is not a single manuscript which is older than 1200 years after Euclid lived, that is about a thousand years ago.

Eight years of work by a young genius standing on the shoulders of great pioneering scholars, have in extraordinary fashion greatly advanced the frontiers of our knowledge of Babylonian mathematics. One can not help feeling that the inspiration of such achievement will cause more than one man to shout, "Let knowledge grow from more to more," as *he* too joins in the endless torch race to

"pass on the deathless brand
From man to man."

THE ART OF TEACHING
A NEW DEPARTMENT

Pupil Formulation of Problems in First Year Algebra

By Dorothy F. Briggs
Northeast Junior High School, Kansas City, Missouri

I have found it helpful, throughout the course in first year-algebra, to have the students formulate problems for themselves, especially for oral practice. Early in the course, as we are developing the idea that, in algebra, a letter represents a number, the pupils ask such questions as "When $n=3$, what is the value of $2n-1$?" or "If $7a=63$, what is the value of a?" Often the pupil stating the problem selects the person to answer it. That student, if answering correctly, asks the next question. Care must be taken, however, to see that opportunity for practice is distributed throughout the class and not monopolized by the more aggressive members.

The same method works well in practice in the fundamental operations with signed numbers. For instance, the class is interested in seeing how many can combine mentally a series of seven or eight similar terms as they are stated orally by a member of the class. The member stating the terms finds that he has a harder task in obtaining the correct result than do those who are working his problem.

In examples in long multiplication and division, after counting the number of chances of making a mistake, it is great fun to formulate a similar problem and see how many can successfully pass over all the existing danger spots.

Equations of simple nature are stated and solved orally. When more difficult equations have been studied, the pupils formulate and solve equations having some designated number for the root.

In the study of special products and factoring the students devise true-false statements which emphasize quite vividly the types of errors which they must avoid.

In problem solution, after oral practice in stating and answering questions concerning the representation of related quantities, the pupils state an equation and then formulate problems leading to that equation, or make a statement about related quantities and then translate the statement into an equation. Often the equation required is beyond their present power of solution.

I find that the course becomes more vivid and more interesting when the students find that algebra problems are not just queer things that only some seemingly mythical and supernatural "author" can concoct, but are simple matters which they can construct perfectly well themselves.

A Short Algebra Program for a P.T.A. Open House Meeting

By Dorothy F. Briggs
Northeast Junior High School, Kansas City, Missouri

In the fall of each year, the P.T.A. of Northeast Junior High School, Kansas City, Missouri sponsors an Open House night meeting, at which, after a short program in the auditorium, each parent follows his child's schedule of classes for fifteen minute periods. In each class period, the "pupils" of the evening have a

chance to meet their "teachers" and to gain some small idea of the courses for which they are "enrolled."

After greeting the patrons as they enter the room, I have about seven or eight minutes remaining for an explanation of what goes on in an algebra class. This explanation always meets with special response when made by the students themselves. This past year the following program was given, with each of my five algebra classes responsible for one of the five talks. Each class chose its own representative after tryouts by volunteers.

1. The Origin of Symbols (The symbols of addition, subtraction, multiplication, division, and equality)
2. An Explanation of a Graph of Our Latest Algebra Test
3. An Original Algebra Problem (Presentation and solution of an original problem based on the attendance, by Home Room Divisions, at the October P.T.A. meeting)
4. A Number Trick Explained by Algebra (After the parents had carried through the steps of a simple number trick, the algebraic explanation of the puzzle was given)
5. What I Think of Algebra after Ten Weeks of the Course

State Representatives of the National Council of Teachers of Mathematics

Mrs. Florence Brooks Miller has announced the appointment of the following persons as official representatives throughout the United States for the Council for 1936 and until further notice.

Alabama: Mr. J. E. Allen, Phillips High School, Birmingham, Alabama

Arizona: None as yet.

Arkansas: Miss Isabella K. Smith, Sr. High School, Fort Smith, Arkansas

California: Miss Emma Hesse, University High School, Oakland, California

Colorado: Mr. H. W. Charlesworth, East High School, Denver, Colorado

Connecticut: Miss Dorothy S. Wheeler, Bulkeley High School, Hartford, Conn.

Delaware: Mr. Edwin H. Downes, Dover, Delaware

District of Columbia: (Divisions 1 to 9) Mr. Wm. J. Wallis, Wilson Teachers College, 11th and Harvard Sts. N.W., Washington, D. C. (Divisions 10 to 13) Mrs. Ethel Harris Grubbs, 751 Fairmont St., N.W., Washington, D. C.

Florida: Mr. F. W. Kokomoor, Univ. of Florida, Gainesville, Florida

Georgia: Miss Viola Perry, Collegeboro, Georgia

Idaho: None as yet.

Illinois: Mr. C. M. Austin, Oak Park, Illinois

Indiana: Mr. L. H. Whitcraft, Ball State Teachers College, Muncie, Indiana

Iowa: Miss Ruth Lane, University High School, Iowa City, Iowa

Kansas: Dr. U. G. Mitchell, Univ. of Kansas, Lawrence, Kansas

Kentucky: Miss Dawn Gilbert, 1331 Clay St., Bowling Green, Kentucky

Louisiana: Mrs. H. L. Garrett, Box 427 University Station, Baton Rouge, Louisiana

Maine: Miss Pauline Herring, 360 Spring St., Portland, Maine

Maryland: Miss Agnes Herbert, Clifton Park Jr. High School, Baltimore, Maryland

Massachusetts: Mr. Harold B. Garland, 129 Houston Ave., Milton, Massachusetts

Michigan: Mr. Duncan A. S. Pirie, 950 Selden Ave., Detroit, Michigan

Minnesota: Mr. H. G. Tiedeman, Mountain Iron, Minnesota

Mississippi: Mr. Dewey S. Dearman, State Teachers College, Hattiesburg, Mississippi

Missouri: Mr. G. H. Jamison, State Teachers College, Kirksville, Missouri

Montana: Miss Gertrude Clark, 403 Eddy Ave., Missoula, Montana

Nebraska: Dr. A. R. Congdon, University of Nebraska, Lincoln, Nebraska

Nevada: None as yet.

New Hampshire: Mr. Merton T. Goodrich, Box 93, Keene, New Hampshire

New Jersey: Mr. Charles D. Wildrick, Englewood, New Jersey
New Mexico: Miss Olive Whitehill, 402 West 8th St., Deming, New Mexico
New York: Mr. H. C. Taylor, Benjamin Franklin H. S., Rochester, New York
North Carolina: Mr. H. F. Munch, Chapel Hill, North Carolina
North Dakota: Miss Henrietta L. Brudos, State Teachers College, Valley City, North Dakota
Ohio: Dr. R. L. Morton, Ohio University, Athens, Ohio
Oklahoma: Miss Kathleen Begley, Tulsa, Oklahoma
Oregon: Mr. Edgar E. De Con, University of Oregon, Eugene, Oregon
Pennsylvania: Dr. C. N. Stokes, Temple Univ., Philadelphia, Pennsylvania
Rhode Island: Mr. W. Wilson Talley, Mary C. Wheeler School, Providence, Rhode Island
South Carolina: Miss Hortense Rogers, Winthrop College, Rock Hill, South Carolina
South Dakota: Miss Agnes Souba, Box 428, Watertown, South Dakota
Tennessee: Dr. F. L. Wren, Peabody College, Nashville, Tennessee
Texas: Miss Elizabeth Dice, North Dallas High School, Dallas, Texas
Utah: Mr. J. A. Gardner, 3679 S. 13th St., Salt Lake City, Utah
Vermont: Mr. G. H. Nicholson, 328 North Ave., Burlington, Vermont
Virginia: Mr. L. G. Lankford, Jr., University of Virginia, University, Virginia
Washington: Miss Kate Bell, The Lewis and Clark High School, Spokane, Washington
West Virginia: Miss May L. Wilt, 107 High St., Morgantown, West Virginia
Wisconsin: Mr. J. S. McNair, Sheboygan High School, Sheboygan, Wisconsin
Wyoming: Miss Ruth Hoag, 1021 S. McKinley St., Casper, Wyoming

The Spirit of Education*

An interview with the artist, N. C. Wyeth

IN THE center of the picture the dense clouds of Ignorance and Prejudice have parted and to the earth has come an heroic figure symbolizing the Spirit of Education. She holds aloft the flaming torch of enlightenment, which we in America are to carry forward from generation to generation, while open before her lies the Book of Learning. Mercury-like she moves over forward on the winged wheel of Progress.

Representative Americans, who typify for their day and time the Spirit of Education, move in slow procession toward the figure, from the right and from the left, and are bathed in the incandescent light from her torch. Epochs in our educational history have been marked by personalities which afforded leadership. Ranged about these leaders are groups of children and young people who, feeling their presence and inspiration, follow them in contentment and utmost faith.

The left-hand processional we might call that of the Pioneer Teachers—pioneer in the sense that these particular leaders ventured into new fields and broke new ground.

Leading this group is an idealized figure of the Colonial schoolmaster, who, as teacher, was largely responsible for the intellectual and spiritual quickening of his generation. Gathered about him are the little boys and girls of his time, carrying their hornbooks.

Next at the left are two Indian boys and a Franciscan priest (Junipero Serra) who symbolizes the heroic work of pioneer missionaries among the Indians, especially on the West Coast.

Directly behind them are two of the Dame school teachers who laid the foundations of our elementary public school system.

Then comes the negro educator, Booker T. Washington, who sought equality for his people in the field of education. With him is a negro boy carrying a hoe to symbolize vocational training.

* See frontispiece in this issue.

At the extreme left is a thoughtful and introspective trio; Benjamin Franklin, representing balanced judgment, Thomas Jefferson, penetrating political and educational sagacity, and Joseph Lancaster, educative originality. All three were active and advanced thinkers in the field of education during the critical and formative period of our nation's history.

In the background at the left side of the mural is the log cabin of the frontier—the land of the pioneers; and, beyond, the long sweep of open country merging into towering mountain peaks—the land of future America.

The right-hand processional is led by the stalwart figure of Horace Mann, one hand protecting the little boy with drooping head across whose eyes is a bandage symbolizing blindness. For among Mann's many contributions to society were his efforts in behalf of instruction for the blind. His work of educating public opinion toward the establishment of common public schools is unique in the history of education. In pledging himself to the task, he said, "I devote myself to the supremest welfare of mankind on earth."

Beside him stands Mary Lyon with two charming girls dressed in the fashion of one hundred years ago. Despite ridicule and criticism her untiring efforts to establish higher education for young women were crowned with success in 1837 when she opened Mount Holyoke Female Seminary.

Then comes Henry Barnard, the philanthropist, next in importance to Horace Mann in the significance of his work for the public schools of America.

With hand raised to direct the singing of a boy and girl is the Beethovenesque figure of the youthful Lowell Mason who was the first to introduce music into American schools.

In scholarly abstraction next stands our contemporary thinker, John Dewey, and behind him Charles W. Eliot, representing the era when the modern university and college were first attaining their present stature. Completing the procession is Colonel Francis W. Parker who influenced educators to keep the child, and not teaching, uppermost in their thinking.

The background at this side of the mural offers a vivid contrast. Rising into the clouds is the skyline of a modern city, flecked with jets of steam. Stacks are belching smoke and up the harbor steams a giant liner. Tugs are puffing by the docks. Here is modern, industrial America.

We have commissioned the artist, Mr. N. C. Wyeth, to paint The Spirit of Education to make the cycle of fifty years in publishing since our company's founding in 1885. We hope that the reproduction of this work of art will be a thought-provoking reminder to all who receive it of our nation's educational heritage and of our national responsibility to "carry on" in the face of all obstacles if our civilization is to survive.

In our own case, we are facing the next half-century with a broadened and more determined idealism, and with hope and confidence of rendering even greater service to the cause of education through the contribution of the textbook.

MATHEMATICS is the queen of the sciences, and the theory of numbers is the queen of mathematics.—EULER

The Status of Mathematics in India and Arabia during the "Dark Ages" of Europe*

By F. W. Kokomoor
University of Florida

Political and economic world historians have found it convenient to divide the time from the fall of Rome to the discovery of America into two periods, and to designate the first of these by the term "Dark Ages." One work accounts for this name by "the inrush into Europe of the barbarians and the almost total eclipse of the light of classical culture." The period covers, roughly, the time from 500 to 1000 A.D. Part of these "barbarians" came down from the north and the rest attacked from the south, the latter bound together politically and religiously by the great, although probably totally illiterate, leader Mohammed into a vast dominion that at one time or another covered all of eastern Asia, northern Africa, Spain, France in part, and the European islands of the Mediterranean Sea. It was during this period that Europe was dark, learning at low ebb, and the development of mathematics almost negligible. The world as a whole was not dark, and as applied to general history the expression "Dark Ages" is a gross misnomer. Throughout the entire period there was considerable intellectual (including mathematical) activity among the Hindus and, beginning about 750, there developed many centers of Muslim civilization which rose to the very peak of mathematical productivity.

A number of facts combine to account for our heretofore slight emphasis upon oriental mathematical science. (1) We have been so enamored by the story of the Golden Age of Greece and the Modern Period that the Orient failed until recently to divert our attention. (2) There is such an overwhelming mass of mediaeval material left us and so much of it is worthless, due largely to the overabundance of study devoted to Scholasticism, that the search for gems among the rubbish has seemed hardly worth the effort. (3) Then, too, the difficulty of accessibility involved is enormous. Before many Greek and Hindu works could be assimilated in the main scientific current in the West, they had to be translated first into Syriac, then Arabic, then Latin and finally into our own language. Thus the completeness and the accuracy of these transmissions can only be determined by painstaking investigations of historians of science. (4) Furthermore, but few persons have ever been qualified to advance our knowledge of Muslim mathematics, due to the rare qualifications required. Not only does one need to know well mathematics and astronomy, but also Sanskrit, Syriac, Arabic and Persian, and, in addition, one needs to have a thorough training in paleography and a keen historical sense. This rare combination, together with a lifetime of ceaseless work, is the price that must be paid to increase our understanding of oriental mathematics.

Modern mathematics is easily transmitted; the process is simple. Articles appearing in any scientific journal are announced in other principal ones, and hence it is easily possible for a worker in any field to be fully informed on what is being done throughout the world regardless of the language used in the original publication. So simple is it that the modern scientist who lacks historical training can hardly comprehend the difficulties involved in the handing down of knowl-

* In the preparation of this article the author has drawn heavily from Dr. George Sarton's *Introduction to the History of Science*, volume I, a work of great value to the student of the history of mathematics.

edge from the early ages to the present.

For our knowledge of Greek mathematics we owe an unpayably large debt to two men who devoted years to the production of numerous accurate translations: J. L. Heiberg of Copenhagen, and T. L. Heath, a great English scholar in both mathematics and Greek. Unfortunately we have neither a Heiberg nor a Heath to enlighten us in Hindu and Muslim mathematics. In our knowledge of the former we are perhaps least fortunate of all. Cantor's three chapters are quite satisfying on the works of Brahmagupta and Bhāskara, being based upon Colebrooke's *Algebra with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhascara*, but later historians place much dependence upon the interpretations of G. R. Kaye who was for many years a resident of India as a high government official, and whose work on Indian mathematics (1915) is shown to be erroneous in many respects by competent scholars of India today, such as Saradakanta Ganguli. One of the ablest scholars in the field of Muslim mathematics was Carl Schoy (1877–1925), who, between the years 1911 and 1925, contributed many valuable papers and books containing critical translations of Muslim mathematics and astronomy.

Recent scholars such as Schoy have shown us that, just as the greatest achievements of antiquity were due to Greek genius, so the greatest achievements of the Middle Ages were due to Hindu and Muslim genius. Furthermore, just as, for many centuries of antiquity, Greek was the dominant progressive language of the learned, so Arabic was the progressive scientific language of mankind during the period of the Middle Ages. We have learned further that the fall of ancient science and the dampening of the scientific spirit in Europe was far less due to the overrunning of southern Europe by the barbarians than it was to the passive indifference of the Romans themselves, and to the theological domination of a little later time. As soon as the Arabs were apprised of the Greek and Hindu sources of mathematical knowledge they were fired with a contagious and effective enthusiasm that led to numerous remarkable investigations in mathematics prosecuted from a number of cultural centers throughout the Muslim world, and that did not abate until the close of the 12th century when they had made a permanent impression on mathematics as a whole.

What I wish to do is to give as comprehensive a survey of Hindu and Muslim mathematics as space permits, pointing out principal achievements of leaders in the field, and indicating work still to be done to make the history complete.

Our attention is first called to Hindu mathematics in one of the five Hindu scientific works on astronomy called *Siddhāntas*, which were theoretical as opposed to karaṇas which were practical. Its date is very uncertain, but is placed in the first half of the fifth century. The *Sūrya-Siddhānta*, the only one we have in full, is composed of fourteen chapters of epic stanzas (ślokas) which show decided knowledge of Greek astronomy but also much Hindu originality, especially the consistent use throughout of sines (jyā) instead of chords, and the first mention of versed sines (utramadjyā).

Even more important is the *Pauliśa Siddhānta* which we have only indirectly through the commentator Varāhamihira (*c.* 505). It contains the foundation of trigonometry and a table of sines and versed sines of angles between 0° and 90° by intervals of 225′ (kramajyā). The sine and the arc of 225′ were taken to be equal, and sines of multiples of 225′ were obtained by a rule equivalent in our symbolism to

$$\sin (n+1)\, x = 2 \sin nx - \sin (n-1)x$$

$$-\frac{\sin nx}{\sin x};\ \sin x = x = 225'.$$

The next important Hindu advance is due to Āryabhaṭa (The Elder) who wrote in 499 a work, *Āryabhaṭīyam*, of four parts,

the second of which—the *Gaṇitapāda*—was a mathematical treatise of 32 stanzas in verse, containing essentially the continued fraction process of solution of indeterminate equations of the first degree; an amazingly accurate value of π, namely 3 177/1250; the solution of the quadratic equation implied in the problem of finding n of an arithmetic series when a, d, and s are known; and the summing of an arithmetic progression after the pth term. These are expressed now by means of the formulas:

$$(1)\qquad n=\frac{d-2a\pm\sqrt{(2a-d)8ds}}{2d}$$

$$(2)\qquad s=n\left[a+\left(\frac{n-1}{2}+p\right)d\right].$$

To him also were due other startling truths less mathematical in character, among which was the theory that the apparent rotation of the heavens is due to the rotation of the earth about its axis.

Varāhamihira, astronomer-poet and contemporary of Āryabhaṭa, contributed the equivalent of trigonometric facts and formulas as follows:

$$\sin 30^\circ = 1/2; \qquad \cos 60^\circ=\sqrt{1-1/4};$$

$$\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}$$

$$\sin^2 x+\text{versin}^2\ x=4\sin^2\frac{x}{2};$$

$$\sin^2 x=\frac{\sin^2 2x}{4}+\frac{[1-\sin(90^\circ-2x)]^2}{4}.$$

Then we have a leap of about a century when Brahmagupta (*c.* 628), one of India's greatest scientists, and the leading scientist of his time of all races, made his study of determinate and indeterminate equations of first and second degree, cyclic quadrilaterals and combinatorial analysis. He solved the quadratic for positive roots completely, and the Pellian equation $nx^2+1=y^2$ in part (it was finished by Bhāskara, 1150). If a, b, c, d, x, and y are the sides and diagonals of a cyclic quadrilateral, s the half-perimeter, and K the area, his results can be expressed by the equations:

$$K=\sqrt{(s-a)\ (s-b)\ (s-c)\ (s-d)};$$

$$x^2=\frac{(ab+cd)\ (ac+bd)}{(ad+bc)};$$

$$y^2=\frac{(ad+bc)\ (ac+bd)}{(ab+cd)}.$$

Another proposition, "Brahmagupta's trapezium," states that if $a^2+b^2=c^2$ and $x^2+y^2=z^2$, then az, cy, bz, cx form a cyclic quadrilateral whose diagonals are at right angles. His work in combinations and permutations is much like that now offered in a first course, but is not quite complete. Brahmagupta used three values of π: for rough work, 3; for "neat" work, $\sqrt{10}$; and for close accuracy, the finer value given by Āryabhaṭa.

By the close of the period of Brahmagupta Hindu mathematics had developed to such an extent that its influence reached out far toward both the east and the west. Two Chinese authors are of special value as witnesses of the influence of Hindu mathematics in China. (1) The first is Ch'ü-t'an Hsi-ta, a Hindu-Chinese astrologer of the first half of the eighth century, whose work gives a detailed account of a number of ancient systems of chronology, the most important of which, from our point of view, is the Hindu system, and in the explanation of which is implied the Hindu decimal notation and rules. Thus we think this must have been the very latest date of the introduction of the Hindu numerals and hence other Hindu mathematics into China. Much more probably they were introduced earlier, about the second half of the sixth century, for the catalog of the Sui dynasty (589–618) lists many books devoted to Hindu mathematics and astronomy. Then, too, it was about this time that Buddhism entered China from India. (2) The second author, I-hsing (683–727), or Chang Sui

(the first being his religious name) was an astronomer of note who undertook, by order of the emperor, an investigation of chronological and arithmetical systems of India, but he failed to finish the work on account of a premature death.

The westward reach of Hindu mathematics is equally certain. The Syrian philosopher and scientist, Severus Sēbōkht (fl. 660), who studied also Greek philosophy and astronomy, was the first to mention the Hindu numerals outside of India, and expressed his full appreciation of Hindu learning in these words: "I will omit all discussion of the science of the Hindus, a people not the same as the Syrians; their subtle discoveries in this science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have reached the limits of science should know these things they would be convinced that there are also others who know something." (Quoted from Smith: *History of Mathematics*, v. I, p. 167.)

But few Hindu mathematicians after the day of Brahmagupta stand out prominently. One of uncertain date but probably of the 9th century is Mahāvīra, author of the *Gaṇita-Sāra-Sangraha*, which deals with arithmetic, including geometric progressions, the relation between the sides of a rational sided right triangle ($2mn$, m^2+n^2, m^2-n^2), and the solution of several types of equations involving the unknown and its square root. Two hundred years later, about 1030, another *Gaṇita-Sāra* (compendium of calculation) was produced by Śrīdhara, but was quite elementary. However, he wrote a work (now lost) on quadratic equations in which, according to the eminent Bhāskara of the 12th century, was found our present formula for the quadratic solution.

Now let us set our clock back at the second half of the 8th century. Here we find that practically all the work done in mathematics was done by Arabs. This is the beginning of a long period extending to the close of the 11th century, during the whole of which there was an overwhelming superiority of Muslim culture. Stimulated from the east by the Hindus and from the west by the eastward transmission of Greek mathematics, the Arabs began a remarkable and altogether too little emphasized flourish of activity. At first there were mainly students and translators of the five Hindu siddhāntas and ohter works. Chief of these were al-Fazārī, the elder, and his son. Then with the 9th century began a series of very important steps forward, especially in the field of trigonometry and the construction of astronomical tables; but there was also an imposing group of geometers, arithmeticians, algebraists and translators of Greek works.

The cause of science was greatly enhanced by the caliph al-Ma'mūn (813–833) who, although religiously exceedingly intolerant, was one of the world's greatest patrons of science. He collected all the Greek manuscripts he could, even sending a special mission into Armenia for that purpose, then ordered the translation of these into Arabic. He built two observatories, had made (probably by al-Khwārizmī) a large map of the world which was a much improved revision of Ptolemy's, organized at Bagdad a scientific academy, and stocked a library which was the finest since the Alexandrian (3d century B.C.). He then invited many of the world's greatest scientists to his court. Among them was al-Khwārizmī (d.*c.* 850) who wrote very important works on arithmetic and algebra and widely used astronomic and trigonometric tables of sines and tangents. He revised Ptolemy's geography, syncretized Hindu and Greek knowledge and is recognized by authorities as having influenced mathematical thought more than any other mediaeval writer.

A very notable astronomer of that period was al-Farghānī (fl. 860) who wrote the first comprehensive treatise in Arabic on astronomy, which was in wide use until the close of the 15th century, and was translated into Latin and Hebrew thus influencing European astronomy greatly before Regiomontanus.

Among the notes and extracts left us from Ḥabash al-Ḥāsib (*c.* 770–*c.* 870) we find record of the determination of time by the altitude of the sun and complete trigonometric tables of tangents, cotangents and cosecants which are at present preserved in the Staatsbibliothek in Berlin. In connection with the problem of finding the sun's altitude, the cotangent function arose. If x = the sun's altitude, h = the height of a stick, and l = the length of its shadow,

$$h=l\left(\frac{\cos\ x}{\sin\ x}\right),$$

and al-Ḥāsib constructed a table of values of h for $x=1,\ 2,\ 3,\ \ldots$ degrees, from which either x or h could be read if the other were known. According to Schoy, the Berlin MS. also contains the equivalent of

$$\sin x=\frac{\tan y \cos z}{\sin z},$$

where y is the declination and z the obliquity of the ecliptic.

One of the world's best general scientists and the greatest philosopher of the Arabs was al-Kindī (fl. 813–842), prolific author of between 250 and 275 works on astronomy, geography, mathematics and physics. He understood thoroughly the Greek mathematical works and influenced widely the early European scientists among whom were Girolamo Cardano, of cubic equation fame, and Roger Bacon.

The Banū Mūsā (Sons of Moses, also known as the Three Brothers) were wealthy scientists and patrons of science of this period. Through their efforts many Greek MSS. were collected, studied, translated and thus preserved for us. They were among the earliest to use the gardener's method of the construction of the ellipse (use of a string and two pins) and discovered a trammel based upon the conchoid for trisecting angles. Various problems of mechanics and geometry interested them most.

Perhaps chief among the many translators is to be mentioned al-Ḥajjāj (*c.* 825) who first (under Hārūn al-Rashīd) translated Euclid's *Elements* into Arabic and improved it later (under al-Ma'mūn) in a second translation. He also was an early translator of Ptolemy's *Syntaxis*, which he called *al-mijisṭi* (the greatest) from which term the name *Almagest* was derived. He was preceded, however, in this work by the Jewish Arab al-Ṭabarī of the same period. By the middle of the 9th century, then, these men and others of their day had made accessible to the Arabs the most important works of the Greeks and the earlier Hindus, had extended the sum total of astronomy and trigonometry, and had given tremendous impetus to independent investigation, which bore its fruit in the century to follow.

There was, however, much translating still to be done. Al-Māhānī (fl. 860) wrote commentaries on at least the first, fifth and tenth books of the *Elements*, and on Archimedes' *Sphere and Cylinder*, and studied also considerably the *Spherics* of Menelaos which led him to an equation of the form $x^3+a^2b=ax^2$, with which he wrestled long enough to cause his successors in the field to refer to it as "Al-Māhānī's equation." He never solved it. Al-Ḥimṣī (d.*c.* 883) translated the first four books of the *Conics* of Apollonius. Al-Nairīzī (d.*c.* 922) wrote (both commentaries lost) most authoritatively on the *Quadripartitum* and *Almagest* of Ptolemy as well as on the *Elements*. Thābit ibn Qurra (*c.* 826–901) founded within his own family a school of translators and enlisted outside scholars to aid him in producing translations of nearly all the Greek mathematical classics even includ-

ing the commentary of Eutocius. Without naming others, suffice it to say that by the beginning of the 10th century there was probably not a single important work of the Golden Age of Greece that had not been translated and mastered by the Arabs.

As to the results of independent investigators of this period, the following must be mentioned. The best and most complete study among the Arabs on the spherical astrolabe was produced by Al-Nairīzī. The school of Thābit ibn Qurra wrote about 50 works of independent research, and 150 (about) translations. Many of these works are still extant. The most valuable ones are those on theory of numbers and the study of parabolas and paraboloids. A sample of the former on amicable numbers shows the fine reasoning employed: $2^n pq$ and $2^n r$ are amicable numbers if p, q, and r are prime to each other and if $p=(3)(2^n)-1$, $q=(3)(2^{n-1})-1$, and $r=(9)(2^{2n-1})-1$. The latter contains ingenious developments of Archimedes' Method.

In many respects the outstanding scholar of the century was al-Battānī (d. 929), Muslim's greatest astronomer, in whose principal work on astronomy are found numerous important facts. He gave the inclination of the ecliptic correct to 6″; calculated the precession at 54.5″ a year; did not believe in the trepidation of the equinoxes, which Copernicus still believed many years later. In trigonometry, to which his fifth chapter is devoted, he gave the equivalent of our formulas:

$$\sin x=\frac{1}{\csc x}; \qquad \cos x=\frac{1}{\sec x};$$

$$\tan x=\frac{\sin x}{\cos x}; \qquad \cot x=\frac{\cos x}{\sin x};$$

$$\csc x=\sqrt{1+\cot^2 x}; \quad \sec x=\sqrt{1+\tan^2 x};$$

$$\sin x=\frac{\tan x}{\sec x};$$

$\cos a=\cos b \cos c+\sin b \sin c \cos A$; and the sine law (doubtful). This work was translated into Latin by Plato of Tivoli in the 12th century and into Spanish the 13th, and exerted a tremendous influence for 500 years.

Considering now the first half of the 10th century, we note again that almost all the original work was done by Arabs in Arabic, but with the marked difference that there is a decided decline in activity. The development of mathematics may be compared with rainfall, which, after uncertain periods of drouth or average intensity, may burst forth in torrents. Some sort of unknown law of rhythm seems to hold rather than a law of uniform advance. The outstanding men of the period are Muslim. Ibrāhīm ibn Sinān (d. 946) wrote numerous commentaries on astronomy and geometry, but has received high recognition only since 1918 when Schoy's translation of his *Quadrature of the Parabola* revealed the fact that his method was superior to and simpler than that of Archimedes. Abū Kāmil (*c.* 925) was an able algebraist, improved essentially the algebra of al-Khwārizmī by some generalizations, development of algebraic multiplication and division, and operations with radicals. He also gave an algebraic treatment of regular inscribed polygons.

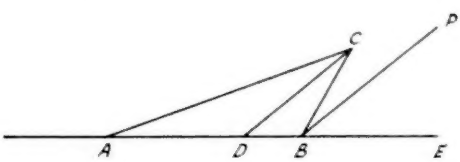
Toward the close of the century a decided renewal of creative work is to be observed. Far more names of famous mathematicians could be mentioned than space permits. But Abū-l-Fatḥ (fl. 982), al-Khāzin (fl. 950), al-Kūhī (fl. 988), al-Sijzī (fl. 1000), al-Ṣūfī (fl. 975), al-Khujandī (fl. 990) and Abū-l-Wafā' (fl. 990) cannot be omitted.

In Florence there is an untranslated commentary on the first five books of Apollonius' *Conics*, by Abū-l-Fatḥ, who also wrote a translation of the first seven books of the *Conics*. His work on books V–VII is considered highly important because the original Greek is no longer extant. Al-Khāzin solved the cubic equation of al-Māhānī (mentioned before in this paper). Al-Kūhī is chiefly known for his

work on the solution of higher degree equations by means of the intersections of two conics. His problem on trisecting an angle I quote here (from French) from Woepcke's *L'Algèbre d'Omar Alkhayyami* (1851), page 118:

Let the given angle be CBE. Take on EB produced points A and D, and on the other side a point C so that (1) $AD=DC$, (2) $AB:BC::BC:BD$. Draw BP parallel to DC. Then the angle $CBP=1/3$ angle CBE.

The geometry of the figure shows clearly that the angle is trisected, but the construction involves the solution of a cubic equation.



Of the works of al-Sijzī, fourteen are now preserved in Cairo, Leyden, Paris, and the British Museum. All deal with conic sections, trisection problems and the resulting cubic equations. These works rank him as an outstanding Arabic geometer.

The greatest mathematician of the century, however, was Abū-l-Wafā', who wrote 15 or 20 works (mostly extant) on geometry, geometrical solutions of special fourth degree equations, and trigonometry, to which he added a number of formulas and the line values of the functions.

The 11th century continues in the first half to show remarkable activity, with an imposing array of first order mathematicians, the principal ones being Al-Bīrūnī (973–1048), Ibn Yūnus (d. 1009), al-Karkhī (d. 1025), Ibn Sīnā (980–1037), al-Ḥusain (?) and al-Nasawī (*c.* 1025). Of paramount importance is the work of Al-Bīrūnī. Two of his writings are of great mathematical significance. We have known the first, *A Summary of Mathematics*, for some time, but the other, *Al-Qānūn al-Masūdī*, has recently been given us in German, and proves its author to be of considerably more importance in the history of trigonometry than we have suspected.

Noteworthy among the contributions of Ibn Yūnus is the introduction of the prosthaphaeretical (sum and difference) formulas of trigonometry which were so useful before the time of logarithms. Al-Karkhī, whose work, *al-Fakhrī*, Woepcke has given us in French, was an algebraist of the first rank. He gave a splendid treatment of the solution of Diophantine equations, equations of quadratic form, operations with radicals, and summation of integro-geometric series. In connection with series he gives such results (not symbolically, of course) as

$$\sum_1^n i^2 = \left(\sum_1^n i\right)\left(\frac{2n+1}{3}\right)$$

and $$\sum_1^n i^3 = \left(\sum_1^n i\right)^2.$$

Ibn Sīnā was principally a philosopher and hence emphasized that phase of mathematics; al-Ḥusain wrote one of the few Arabic treatises on the construction of right triangles with rational sides; and al-Nasawī, an able arithmetician, explained extraction of square and cube roots by a method very similar to our own, and furthermore anticipated decimal fractions in the manner indicated in the equations

$$\sqrt{17} = \frac{1}{100}\sqrt{170000} = \frac{412}{100},$$

but he changed them to sexagesimals for the final form of his answer.

There is a distinct decline toward the close of the 11th century with a notable decrease in the number of mathematicians of the first rank. Of these I mention but one, Omar Khayyan, who, although living in a period of decline, was one of the greatest of the Middle Ages. His chief distinction results from his admirable work on *Algebra*, in which he classified equations by the number and degree of terms, treated 13 types of cubic equations, and

referred to the general expansion of the binomial with positive integral coefficients which he treated in another work now unknown. His other mathematical writings dealt with the assumptions of Euclid and a very accurate reform of the calendar.

Thus closes the marvelous periods of scientific activity of the Hindus and Muslims in the field of mathematics. The Hindu period began shortly after the time of Proclus (d. 485), last straggler of the great Greek period. After the Hindu peak comes the tremendous Muslim flourish. Suter in his work, *Die Mathematiker und Astronomen der Araber und ihre Werke*, lists the works of 528 Arabic scholars who were active in mathematics from 750 to 1600. Certainly 400 of these came within the period of the "Dark Ages." We have learned much about them, but there is still much to be done. Tropfke tells us that "ueberreiche Schaetze" still lie untranslated in the large libraries of Europe.

During this same period in Europe, but few names deserve even feeble mention as writers on mathematics. There was no creative work. Boetius (d. 524) wrote texts on the quadrivium (arithmetic, music, geometry and astronomy), which, though comparatively poor, were so widely used in schools that they had tremendous influence. Anthemios (d. 534) is interesting for his history of conic sections and his use of the focus and directrix in the construction of the parabola. Eutocius (b. 480?) is important for his commentaries on the works of Archimedes and Apollonius. Bede (673–735) deserves mention because our information on finger reckoning is almost entirely dependent on his work. Alcuin or Albinus (735–804) wrote a work on puzzle problems which furnished material for textbook writers for ten centuries. Gerbert (Pope Sylvester II) (999) was great and influential as compared with other European writers of his day and doubtless did much to popularize the Hindu-Arabic numerals. But these men were all small as compared with the Muslim giants.

Let me repeat what I said at the beginning: Europe was dark, but India and the Muslim world were not. To quote Dr. George Sarton, "the 'Dark Ages' were never so dark as our ignorance of them."

Does the harmony which human intelligence thinks it discovers in Nature exist apart from such intelligence? Assuredly no. A reality completely independent of the spirit that conceives it, sees it, or feels it, is an impossibility. A world so external as that, even if it existed, would be forever inaccessible to us. What we call *objective reality* is, strictly speaking, that which is common to several thinking beings and might be common to all; this common part, we shall see, can only be the harmony expressed by mathematical laws.—Poincaré. *The Value of Science.*

There is no subject in the entire high school curriculum which by its very nature lends itself more admirably to a realization of some educational objectives as does the teaching of mathematics. I know of no other subject, save possibly foreign language, where a close day by day application of the student is so absolutely essential to success. When can the boy or girl be more impressed with the importance of doing his job day by day, and meeting his obligations that lead to his own success than in a class of mathematics? He is constantly confronted with the necessity of being alert, critical and observant. He learns to develop habits of inspection and inquiry concerning printed and spoken statements, habits worth cultivating by everyone today. He learns to take nothing for granted except certain hypotheses about which there can be no argument. He waits for all the evidence to come into the picture before drawing conclusions. He finally accepts no opinion, theory or notion that is not backed by facts. Empirical reasoning is soon detected and labeled as such.

Geometry is a powerful training in logic. When Lincoln had a difficult case to try in court he resorted to Euclid as the most helpful aid to jurisprudence. The practical value of geometry was much larger to Lincoln than its mere application to the arts and industries. From an article by S. W. Lavengood, Principal, Pershing School, Tulsa, Oklahoma in the Georgia Educational Journal, Feb. 1936, p. 27 on "Contributions of the Teaching of Mathematics."

Improving Ability in Fractions

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During the school year 1932–33 Donald Garrison conducted under the direction of the writer a diagnostic and remedial project in fractions with the sixth-, seventh- and eighth-grade pupils in the Franklin Township Centralized School, Darke County, Ohio. The procedure utilized and the results attained are reported in this article. The project extended over a period of twenty-four weeks, utilizing one forty-five minute period per week. The actual time devoted to the remedial work did not exceed eighteen hours.

Discovering the pupils weak in fractions constituted the first step in the project. For this purpose the Guiler-Christofferson Diagnostic Survey Test in Computational Arithmetic, Form 1, was used. This test covers the following units: whole numbers, fractions, decimals, practical measurements, and percentage. The test was given to the eighty-eight pupils enrolled in grades six, seven and eight. Of this number, fifty-eight fell below the standards set for their respective grades on the part of the test which deals with fractions. The remedial program was limited to these weak pupils. Their average intelligence ratings are shown in Table I.

Diagnosing and recording group and individual needs constituted the second and third steps, respectively, in the remedial program. Pupil needs were diagnosed by means of the Guiler-Christofferson Diagnostic Test in Fractions, Form 1. This test covers five phases of fractions: vocabulary and manipulation, addition, subtraction, multiplication, and division. Five abilities are included in each part of the test, and two examples are used to measure each ability, the examples being arranged in cycle form. Each example has a value of one point, the highest possible score being fifty points. The particular abilities measured by the diagnostic test are presented in Table III, and the scores of the fifty-eight pupils on the test are given in Columns 1, 3, 5, and 7 of Table II.

An examination of the preliminary diagnostic test scores presented in Table II reveals a number of interesting facts. First, the average total scores show steady progress from grade to grade; however, more progress was made from grade six to grade seven than was made from grade seven to grade eight. Second, there was wide variation in achievement. In the sixth grade, for example, the highest total score was almost four times as large as the lowest total score. Third, there was much overlapping in achievement. Thus, more than one-fourth ($26\frac{2}{3}\%$) of the seventh-grade pupils made higher total scores than those attained by more than one-half ($58\frac{1}{3}\%$) of the eighth-grade pupils; one-fourth of the sixth-grade pupils made higher total scores

Table I

*Intelligence Ratings of Remedial Pupils**

	Number of Pupils	Mental Age		Intelligence Quotient	
		Average	Range	Average	Range
Grade 6	16	11 yr. 4 mo.	9–11–13–4	84.6	70–108
Grade 7	30	12 yr. 8 mo.	10–9–14–10	92.9	72–121
Grade 8	12	13 yr. 10 mo.	12–9–15–7	88.8	76–107
All Three Grades	58	12 yr. 6 mo.	9–11–15–7	89.8	70–121

* Ratings were derived from the use of the Detroit Alpha Intelligence Test.

TABLE II

Total Scores on Preliminary and Final Tests in Fractions

Total Score	Grade 6		Grade 7		Grade 8		All Three Grades	
	(1) Pre-test	(2) Re-test	(3) Pre-test	(4) Re-test	(5) Pre-test	(6) Re-test	(7) Pre-test	(8) Re-test
49				2		2		4
48				3				3
47				2		2		4
46		1		1		2		4
45				3	1		1	3
44				2				2
43		2		3		1		6
42				2				2
41		2		1				3
40		2	1	4		1	1	7
39		1	1				1	1
38		2	2	1	1		3	3
37			1	1	1	2	2	3
36		2		1				3
35			1	1	1	2	2	3
34	1		2	1	1		4	1
33	1	1			1		2	1
32			2		1		3	
31	1			1			1	1
30	1	1	6	1	1		8	2
29	1		1		1		3	
28		1	1		1		2	1
27	1		1				2	
26	1						1	
25	1		2		1		4	
24	2		1				3	
23	3		3				6	
22			1				1	
20			2		1		3	
19	1	1	1				2	1
18	1						1	
17			1				1	
9	1						1	
Total	16	16	30	30	12	12	58	58
Average	24.9	36.9	28.8	42.0	32.2	42.6	28.4	40.7

than those attained by one-third of the eighth-grade pupils.

The major needs and the specific needs of the various pupil groups are clearly indicated by the error quotient data for the pre-test presented in Table IV. Error quotients are "determined by using the frequencies of error for an individual or for a group as a numerator of a fraction in which the denominator shall represent chances for error."[1] Reference to the top figure in the first column will serve to show how the error quotients were computed. Thus, sixteen pupils in the sixth grade were included in the preliminary diagnostic test. Since two examples were used to measure each of the abilities included in the test, there were 32 (2×16) chances for this group to obtain wrong answers to the examples used for measuring Ability 1. The number of wrong answers actually obtained by the group was 26; hence, the error quotient was .81 ($26\div32$).

A number of significant facts are revealed by an analysis of Table IV. One fact is that marked variation character-

[1] Stormzand, Martin J., and O'Shea, M. V., *How Much English Grammar?* Baltimore: Warwick and York, Inc., 1924, p. 14.

TABLE III

*Abilities in Which Eighth-Grade Pupils Were Weak**

Abilities in Which Pupils Were Weak	Pupils												Number of Pupils Mani-festing Weakness
	1	2	3	4	5	6	7	8	9	10	11	12	
PART I—VOCABULARY AND MANIPULATION													
1. Recognition of meaning of terms	x	x	x	x	x	x	x	x	x	x		x	11
2. Reducing a fraction to lowest terms													
3. Changing a fraction to an equivalent fraction having a given denominator	x							x					2
4. Changing an improper fraction to a mixed number					x	x				x			3
5. Changing a mixed number to an improper fraction	x				x		x						3
PART II—ADDITION													
6. Fractions and mixed numbers with like denominators (no carrying)		x			x		x	x		x			5
7. Fractions and mixed numbers with unlike denominators, the least common denominator being one of the given denominators (no carrying)	x	x			x			x	x	x	x	x	8
8. Fractions and mixed numbers with unlike denominators, the least common denominator being the product of the given denominators (no carrying)	x	x			x	x		x		x			6
9. Fractions and mixed numbers with unlike denominators, the least common denominator being less than the product of the given denominators (no carrying)	x	x	x		x	x	x	x	x	x		x	10
10. Two or more fractions or mixed numbers (carrying)	x		x	x	x	x	x	x	x	x	x		10
PART III—SUBTRACTION													
11. Fractions and mixed numbers with like denominators (no borrowing)	x				x				x	x			4
12. Fractions and mixed numbers with unlike denominators, the least common denominator being one of the given denominators (no borrowing)	x					x	x	x	x			x	6
13. Fractions and mixed numbers with unlike denominators, the least common denominator being the product of the given denominators (no borrowing)	x		x		x		x	x	x	x	x		8
14. Fractions and mixed numbers with unlike denominators, the least common denominator being less than the product of the given denominators (no borrowing)	x	x				x		x		x			5
15. Fractions and mixed numbers (borrowing)	x				x		x		x	x	x		6
PART IV—MULTIPLICATION													
16. Fraction by an integer or an integer by a fraction			x		x	x		x	x	x	x		7
17. Mixed number by an integer or an integer by a mixed number			x		x		x	x	x	x	x		7
18. Fraction by a fraction	x		x		x	x		x	x		x	x	8
19. Mixed number by a mixed number			x		x		x	x	x	x			6
20. Three or more fractions or mixed numbers when cancellation is used	x	x	x	x	x	x	x	x	x	x	x	x	12

* As revealed by the Preliminary Diagnostic Test.

TABLE III (*Continued*)

Abilities in Which Pupils Were Weak	Pupils												Number of Pupils Manifesting Weakness
	1	2	3	4	5	6	7	8	9	10	11	12	
PART V—DIVISION													
21. Fraction or a mixed number by an integer		x			x			x	x		x		5
22. Integer by a fraction	x		x		x	x	x	x	x	x	x	x	10
23. Fraction or a mixed number by a fraction	x		x	x	x	x		x	x		x		8
24. Integer by a mixed number	x	x	x		x		x	x	x	x	x	x	10
25. Fraction or a mixed number by a mixed number	x	x	x	x	x	x	x	x	x	x	x	x	12
Number of Abilities in Which Pupils Were Weak	18	10	13	5	21	13	14	20	18	18	13	9	172

ized the extent to which fractions had been mastered by the three groups of pupils. This fact is manifested in several ways. First, the three pupil groups varied considerably in their mastery of the entire unit, the error quotients being .50 for the sixth grade, .42 for the seventh grade, and .36 for the eighth grade. Second, the three pupil groups showed marked variation in their mastery of certain phases of the unit. In vocabulary and manipulation the error quotient was .46 for grade six, .32 for grade seven, and .18 for grade eight; in subtraction and in multiplication, respectively, the corresponding error quotients were .46, .41, .28 and .62, .50, .43. The least variation among the three pupil groups in mastery of the five phases of fractions occurred in addition and in division in the order named. A second fact is that the different pupil groups were much weaker in certain phases of fractions than in others. All three groups encountered more difficulty in multiplication and in division than in the other three phases of the unit. Likewise, the different pupil groups were weaker in certain abilities than in others. The error quotient was .50 or above for twelve abilities (1, 5, 9, 10, 14, 15, 17, 19, 20, 23, 24 and 25) in grade six, for eight abilities (1, 10, 17, 19, 20, 23, 24 and 25) in grade seven, and for five abilities (1, 20, 22, 24 and 25) in grade eight; on the other hand, the error quotient was .25 or below for six abilities (2, 3, 6, 7, 11 and 12) in grade six, for three abilities (2, 3 and 6) in grade seven, and for ten abilities (2, 3, 4, 5, 6, 8, 11, 12, 15 and 21) in grade eight. A third fact is that in the addition phase of fractions and in certain specific abilities more difficulty was encountered by a pupil group than was experienced by a pupil group of lower grade. For Abilities 1, 7, 9, 18, 20, 22 and 25 the error quotient was higher in grade eight than in grade seven; for Abilities 2, 6, 7, 11, 12, 13, 21 and 22 the error quotient was higher in grade seven than in grade six; and for Abilities 6, 7, 11, 13 and 22 the error quotient was higher in grade eight than in grade six.

Individual needs were discovered by an analysis of the pupils' test papers, and diagnostic charts were made showing the weaknesses of each pupil. The individual learning needs resulting from this type of analysis for eighth-grade pupils are shown in Table III. Several salient facts appear in this table. First, the twelve eighth-grade pupils varied greatly in the number of abilities in which they exhibited weakness. Thus, Pupil 8 was found weak in four times as many abilities as Pupil 4. Second, much more difficulty was encountered with certain abilities than with others. In four of the twenty-five abilities covered by the preliminary diagnostic test weakness was not shown by more than one-fourth of the pupils; on the other hand, in seven abilities, weakness was shown by more

TABLE IV

Error Quotients of Sixth-, Seventh-, and Eighth-Grade Pupils (a) on Major Divisions of the Preliminary Test in Fractions and (b) on the Specific Abilities Measured by the Test

Major Divisions and Specific Abilities Covered by the Preliminary Test*	Error Quotients†							
	Grade 6		Grade 7		Grade 8		All Three Grades	
	Pre-test	Re-test	Pre-test	Re-test	Pre-test	Re-test	Pre-test	Re-test
PART I—VOCABULARY AND MANIPULATION								
1	.81	.25	.53	.05	.54	.12	.61	.12
2	.09	.06	.17	.10	.00	.00	.11	.07
3	.25	.19	.25	.07	.08	.04	.22	.09
4	.34	.09	.33	.02	.13	.12	.29	.06
5	.78	.16	.33	.20	.17	.04	.42	.16
Average for Part I‡	.46	.15	.32	.09	.18	.07	.33	.10
PART II—ADDITION								
6	.09	.12	.23	.08	.21	.00	.19	.08
7	.13	.19	.32	.13	.38	.08	.28	.14
8	.47	.28	.28	.07	.25	.04	.33	.12
9	.56	.28	.27	.27	.46	.25	.39	.27
10	.69	.41	.57	.25	.46	.33	.58	.31
Average for Part II	.39	.26	.33	.16	.35	.14	.35	.18
PART III—SUBTRACTION								
11	.06	.12	.27	.12	.17	.04	.19	.10
12	.25	.19	.40	.10	.25	.17	.33	.14
13	.41	.19	.47	.13	.46	.08	.45	.14
14	.75	.28	.45	.25	.29	.12	.50	.23
15	.81	.56	.48	.28	.25	.29	.53	.36
Average for Part III	.46	.27	.41	.18	.28	.14	.40	.19
PART IV—MULTIPLICATION								
16	.38	.09	.33	.03	.29	.00	.34	.04
17	.50	.34	.50	.07	.29	.08	.48	.15
18	.44	.34	.35	.17	.42	.08	.39	.20
19	.75	.41	.52	.40	.33	.29	.54	.38
20	.94	.66	.68	.33	.82	.25	.84	.41
Average for Part IV	.62	.37	.50	.20	.43	.14	.52	.23
PART V—DIVISION								
21	.31	.22	.37	.13	.25	.17	.33	.16
22	.41	.28	.48	.10	.50	.29	.47	.19
23	.53	.22	.53	.20	.46	.25	.52	.22
24	.78	.19	.63	.27	.58	.25	.66	.24
25	.94	.41	.75	.18	.88	.29	.83	.27
Average for Part V	.59	.26	.55	.18	.53	.25	.56	.22
Average for entire test	.50	.26	.42	.16	.36	.15	.43	.19

* See Table III for statement of specific abilities.

† The error quotient was found by dividing the number of wrong answers by the number of opportunities to obtain the wrong answers.

‡ All averages were computed from original data.

than three-fourths of the pupils. None of the pupils were weak in Ability 2; however, all of the pupils were weak in Abilities 20 and 25. Third, the pupils manifested marked individuality in the abilities in which they were weak. Thus, while Pupils 6 and 11 each manifested weakness in thirteen abilities, they were weak in only seven of the same abilities.

The fourth step in the program consisted in giving remedial instruction to overcome the difficulties encountered by

the pupils in the preliminary diagnostic test. At this point the diagnostic charts, one of which is reproduced in Table III, proved very helpful. The remedial work was organized as individualized group instruction, and grade lines were disregarded. In the case of those abilities in which a majority of the pupils were weak group instruction was employed. When only a limited number of pupils were deficient in a given ability, instruction was organized for the particular pupils concerned. The instruction consisted in direct teaching by means of numerous examples and in an abundance of purposeful practice. Individualization of the remedial work was made possible through the use of *A Mastery Work-Book in Fractions*[2] in which the exercises are organized in such a way that each pupil secures extensive practice on the abilities in which he is weak.

The final step in the remedial program consisted in measuring the results that were attained. For this purpose the Guiler-Christofferson Diagnostic Test in Fractions, Form 2, was used. This test is the equivalent of the preliminary test in content and in difficulty. Tables II, IV, V, VI, and VII show the results that were obtained.

One outstanding fact revealed by an analysis of these tables is that marked improvement was made in ability to manage fractions by all the pupil groups included in the remedial project. Improvement in terms of test scores is recorded in Tables II and V. An examination of these tables shows that striking gains in total point score were made by each of the pupil groups; moreover, consistent and significant gains were made by all of the grade groups in each of the five phases of fractions covered by the tests. The percentage of gain in total point score from the preliminary test to the final test was 48.0 for sixth-grade pupils, 62.3 for seventh-grade pupils, 58.4 for eighth-grade pupils, and 57.0 for pupils in all three grades combined. Reference to the percentage of improvement made by the sixth-grade pupils on all five parts of the fractions unit will serve to show how the computation was actually made. Thus, there were sixteen pupils in the group, and the highest possible score was 50 points.

TABLE V

Pupil Improvement in Terms of Test Scores

	Part I	Part II	Part III	Part IV	Part V	All Five Parts
Average Score* on Initial Test						
Grade 6	5.4	6.1	5.4	3.8	4.1	24.9
Grade 7	6.8	6.7	5.9	5.0	4.5	28.8
Grade 8	8.2	6.5	7.2	5.7	4.7	32.2
All grades	6.7	6.5	6.0	4.8	4.4	28.4
Average Score on Final Test						
Grade 6	8.5	6.8	7.3	6.3	7.4	36.9
Grade 7	9.1	8.4	8.2	8.0	8.2	42.0
Grade 8	9.3	8.6	8.6	8.6	7.5	42.6
All grades	9.0	8.2	8.0	7.7	7.8	40.7
Percentage† of Improvement						
Grade 6	67.1	33.9	41.1	40.4	55.8	48.0
Grade 7	73.2	52.0	57.3	59.7	68.1	62.3
Grade 8	63.6	59.5	50.0	67.3	53.1	58.4
All grades	69.8	48.0	51.1	54.7	61.5	57.0

* All averages were computed directly from the test scores.

† The percentage of improvement was computed from original data and was found by dividing the actual gain in point score by the possible gain in point score.

[2] H. C. Christofferson, Carmille Holley, and Walter Scribner Guiler, *A Mastery Work-Book in Fractions*. Philadelphia: F. A. Davis Company, 1932.

Hence, the pupils in this group might have made a total score of 800 points (16×50). Their total score on the initial test was 398 points and on the final test 591 points. The possible gain for the group from initial to final test was 402 points (800−398). The actual gain was 193 points. Accordingly, the percentage of improvement was 48.0 (193÷402). When individual scores are considered (Table II), it is found that 75% of the sixth-grade pupils, $63\frac{1}{3}\%$ of the seventh-grade pupils and 50% of the eighth-grade pupils made scores in the final test that were higher than the best grade score attained by any of the same pupils in the preliminary test.

Data bearing on the number and percentage of abilities in which the different pupil groups manifested weakness in the initial and final tests are presented in Table VI. A study of this table shows that the remedial program was quite effective in reducing the number of abilities with which the various pupil groups encountered difficulty. The average number of abilities in which weakness was found in the unit as a whole was reduced from 16.9 to 10.2 for sixth-grade pupils, from 16.5 to 7.2 for seventh-grade pupils, from 14.3 to 6.1 for eighth-grade pupils, and from 16.2 to 7.8 for pupils in all three grades combined. The decreases in number of abilities in which the pupils were deficient represented percentage decreases of 39.6 in grade six, 56.7 in grade seven, 57.6 in grade eight, and 51.9 in all three grades combined.

TABLE VI

Pupil Improvement in Terms of Abilities Attained

	Part I	Part II	Part III	Part IV	Part V	All Five Parts
Average* Number of Abilities in Initial Test in Which Pupils Were Weak						
Grade 6	3.0	3.0	3.1	3.9	3.8	16.9
Grade 7	2.8	2.9	3.2	3.7	4.0	16.5
Grade 8	1.6	3.2	2.4	3.3	3.8	14.3
All grades	2.6	3.0	3.0	3.7	3.9	16.2
Average Number of Abilities in Final Test in Which Pupils Were Weak						
Grade 6	1.2	2.2	2.0	2.7	2.0	10.2
Grade 7	.7	1.5	1.6	1.7	1.6	7.2
Grade 8	.6	1.2	1.2	1.1	1.9	6.1
All grades	.8	1.6	1.7	1.9	1.8	7.8
Percentage† of Decrease from Initial to Final Test in Number of Abilities in Which Pupils Were Weak						
Grade 6	58.3	27.1	36.0	30.1	47.5	39.6
Grade 7	73.8	48.3	48.4	53.6	60.0	56.7
Grade 8	63.2	61.5	48.3	67.5	48.9	57.6
All grades	67.5	45.4	44.8	49.3	54.4	51.9

* Averages were computed from original data.
† Percentages were computed from original data.

Error quotient data for the three pupil groups included in the remedial project are presented in Tables IV and VII. An analysis of these tables shows that the error quotient in all three grades was materially reduced from initial to final test not only for the unit as a whole but also for the various parts of the unit and for specific abilities. The error quotient for the total

unit was reduced from .50 to .26 in grade six; from .42 to .16 in grade seven; from .36 to .15 in grade eight; and, from .43 to .19 in all three grades combined.

Another significant fact revealed by the test data is that wide individual differences characterized the amount of improvement that was made. This fact is shown in two ways. First, there were wide differences in the amount of improvement in grades seven and eight. Second, there was marked variation in the improvement made by individual pupils in the same grade. Examination of the test scores shows that certain seventh-grade pupils, for example, made a gain in point score which was four times that made by other pupils in the same grade.

A third important fact revealed by the test results is that certain phases and elements of the fractions unit presented more learning difficulties than did other phases and elements. Analysis of Tables IV and VII shows that the error quotient in the final test for all three grades combined was lower in vocabulary and manipulation and higher in multiplication and division than in the other phases of the unit. The data show, likewise, that certain abilities in fractions are more difficult to develop than are other abilities. In this connection, the error quotients of all three grades combined on the final test (Table IV) may be noted for Abilities 10, 15, 19 and 20 as compared with those for Abilities 2, 3, 4, 6, 11 and 16.

TABLE VII

Pupil Improvement in Terms of Error Quotients

	Part I	Part II	Part III	Part IV	Part V	All Five Parts
Average* Error Quotient on Initial Test						
Grade 6	.46	.39	.46	.62	.59	.50
Grade 7	.32	.33	.41	.50	.55	.42
Grade 8	.18	.35	.28	.43	.53	.36
All grades	.33	.35	.40	.52	.56	.43
Average Error Quotient on Final Test						
Grade 6	.15	.26	.27	.37	.26	.26
Grade 7	.09	.16	.18	.20	.18	.16
Grade 8	.07	.14	.14	.14	.25	.15
All grades	.10	.18	.19	.23	.22	.19
Decrease in Size of Average Error Quotient from Initial to Final Test						
Grade 6	.31	.13	.19	.25	.33	.24
Grade 7	.23	.17	.23	.30	.37	.26
Grade 8	.11	.21	.14	.29	.28	.21
All grades	.23	.17	.21	.29	.34	.24

* Averages were computed from original data.

attained by the three groups of pupils. An examination of the data bearing on total test scores (Table V) and on decrease in error quotients (Table VII) shows that the greatest percentage of improvement was made by the seventh-grade pupils. The fact that the seventh-grade pupils attained practically the same average retest score as that attained by the eighth-grade pupils is probably explained by the higher average intelligence quotient of the seventh-grade pupils. When improvement is considered in terms of abilities attained (Table VI), it is found that the least percentage of gain was made by the pupils in grade six and that approximately the same percentage of gain was made by the pupils

A final revelation of the test data is that

several abilities presented unusual difficulty for the sixth-grade pupils. An inspection of Table IV shows that on the final test the error quotient for sixth-grade pupils was above .40 for five abilities, above .50 for two abilities, and above .60 for one ability.

One interesting aspect of the remedial project which should be mentioned in passing relates to the effect of intelligence on developing ability in the use of fractions. One fact which shows that intelligence does exert a great deal of influence in this respect is the degree of relationship which was found to exist between mental age and scores on the initial and final tests. The correlations between mental age and scores on the initial and final diagnostic tests were $.68 \pm .05$ and $.67 \pm .05$ respectively; the correlations between intelligence quotients and scores on the initial and final diagnostic tests were $.50 \pm .07$ and $.66 \pm .05$ respectively. A second fact which indicates the effect of intelligence on the learning of fractions is that the average scores on both the initial and the final diagnostic test were progressively lower (Table VIII) from the best mental age group to the poorest.

The following statements, which are made by way of summary and conclusion, are based on the data that have been presented.

1. Mastery of fractions involves many specific but integrated reactions.

2. The pupils within each grade varied greatly in their mastery of the field and in their mastery of specific uses of fractions.

3. Wide individual differences characterized the amount of improvement that was made.

4. More difficulty was encountered with certain phases and elements in fractions than with other phases and elements.

5. Mental age was found to exert marked influence on ability to learn fractions. The influence of intelligence quotient on mastery of fractions was quite marked although it was less potent and less consistent than that of mental age.

6. Marked improvement in ability to use fractions may be expected from a remedial program which first discovers individual weaknesses and then provides instruction and practice definitely suited to pupil needs.

Table VIII

Average Scores on Initial and Final Tests of Pupils Grouped According to (a) Mental Age (b) Intelligence Quotient

Intelligence Groups	Average Score on Diagnostic Tests of Pupils Grouped According to Mental Age			Average Score on Diagnostic Tests of Pupils Grouped According to Intelligence Quotient		
	Average Mental Age	Average Pre-test Score	Average Re-test Score	Average Intelligence Quotient	Average Pre-test Score	Average Re-test Score
Highest Quarter	14–4	33.9	45.3	107.4	34.5	46.9
Second Quarter	13–1	30.8	42.9	93.2	28.6	40.5
Third Quarter	12–0	25.8	40.4	84.2	27.4	42.1
Lowest Quarter	10–11	23.4	35.5	75.2	23.5	34.7
Upper Half	13–8	32.3	44.0	100.0	31.4	43.6
Lower Half	11–5	24.6	37.9	79.6	25.4	38.3

Do not forget the Summer Meeting of the National Council of Teachers of Mathematics at Portland, Oregon, June 27 to 29. See the program on page 256.

Generalization as a Method in Teaching Mathematics[1]

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Introduction

There is an extensive and rapidly growing literature on the teaching of high school mathematics. It is not the purpose of the present paper to comment on the sins of specific authors, but it may be remarked in passing that many of the current offerings seem inordinately dilute and that numerous doctrines which are exploited as modern and novel appear on closer scrutiny to be old familiar friends decked out in false whiskers and new wigs. One notes too the profusion of those terrible plurals so characteristic of recent pedagogical writing: "attitudes," "skills," "techniques," "procedures," "outcomes," "appreciations," even "understandings."

Much of the emphasis on particular methods is inspired by the effort to adjust the instruction to the accepted pedagogical fashion. Every change in educational philosophy is reflected in the class room. For the enterprising and progressive superintendent feels that he must keep up with the procession. In my own life time I have witnessed pedagogical epidemics sweep over the country as numerous, and sometimes as virulent, as the plagues which Moses visited upon the Egyptians. Who cannot remember when we were all zealously engaged in teaching for character, for citizenship, for efficiency, for democracy, for social adjustment, for the wise use of leisure? The president of a certain college some years ago announced as the purpose of his institution the preparation of students for life twenty years after graduation, while not a few colleges have held the proper goal of education to be the preparation for life eternal. We all recall the Speer method, the Perry movement, the Montessori system, the project method, the Dalton plan, the platoon system, the contract system—the gospel according to Morrison.

Most of the popular modes have erred in attempting to concentrate the complex educational processes under a single aim. At the other extreme is the current tendency to unduly multiply the educational aims by microscopic subdivision. Thus Breslich[2] in a list that is "by no means complete" actually exhibits *one hundred and sixty-two* objectives in the teaching of high school mathematics—almost one for each day of the school year. Fancy the frail teacher of mathematics marching into her class room each morning armed with 162 definite objectives, to be realized for each of 40 pupils! And imagine her sense of humiliation and defeat at the close of the year when she is forced to admit that with respect to 10% of the class she had fallen short by some 37.

I was somewhat surprised too at the distribution of the 162 objectives under the six major headings of powers, appreciations, understandings, attitudes, habits and ideals, and skills. Thus while only 6 "skills" and 16 "understandings" are rated as desirable of attainment, the author recognizes no fewer than 21 commendable "attitudes" and 28 significant "appreciations." Shall we insist on 21 attitudes when the Great Teacher himself proposed only 9 beatitudes—which are essentially attitudes carrying a reward? I would be willing to trade the entire 21 for a single attitude, could it be attained: Blessed are they which do hunger and thirst after mathematics—and I should be wholly

[2] The Teaching of Mathematics in Secondary Schools. Vol. I, pp. 203–8.

[1] An address given to the Mathematics Section of a regional meeting of the Washington Education Association.

content with the teacher who could supply the promised reward: For they shall be filled. If the high school graduates who enter the universities appear somewhat deficient in their mathematical preparation, I venture to ask whether this might not possibly be due to the over-emphasis which some of the schools are placing on "attitudes" and "appreciations," to the consequent neglect of "skills" and "understandings."

In examining books and articles on the teaching of mathematics I am impressed with the prominence that most writers give to the technique of instruction: Class room procedure, the organization of the subject matter, special devices for the attack upon particularly troublesome topics such as the law of signs in algebra, setting up an equation or formula to fit a verbal description, solution of original exercises in geometry, methods of stimulating interest, providing for individual differences, preparing and administering tests, etc. The teacher indeed must attend to all of these details. But many of the special methods seem to me to be largely artificial, grafted upon the subject, as it were. In sharp contrast to these are certain *natural* or *inherent* methods—methods suggested by the nature, structure, spirit, in fine the genius of the subject itself. I refer to the methods employed by the masters in developing our varied, rational and beautiful discipline which the great Gauss so happily called the Queen of the Sciences. Such for example are induction, inversion,[3] generalization, correspondence, transformation—all exercised with the constant interplay of the reason and imagination. To these might be added (a) the scientific method in general, which mathematics shares with the natural and physical sciences and (b) the axiomatic approach in combination with deductive logic, illustrated by demonstrative geometry. The inductive method is perhaps sufficiently exemplified in the solution of original exercises which involve the proofs of theorems. While the original constructions and locus problems require a blending of both the deductive and inductive methods.

Generalization in Algebra

Each great branch of mathematics provides its own peculiar and appropriate methods.[4] But there is one method, namely generalization which permeates the whole science. This is a powerful weapon which has helped to extend the frontier of mathematical knowledge in every direction, and indeed has not infrequently led to the creation of new branches. I propose to consider here how this method may be effectively capitalized in teaching.

We need not look far to see the influence of generalization even in the beginning of our subject. Thus a large part of arithmetic consists in extending to common and decimal fractions the fundamental operations of addition, subtraction, multiplication and division, which were first learned for the integers. Indeed the very concept of the fraction is the first step in the generalization of number. The next step was the introduction of irrationals such as $\sqrt{2}$ which is necessary if every integer is to be the square of some number. A comparatively late event was

[3] By inversion I mean the process by which inverse operations are obtained from one another. Pairs of inverse operations occur everywhere in mathematics: addition and subtraction, multiplication and division, raising to powers and extracting roots, differentiation and integration, rotations in opposite senses, etc.

[4] Or as Whitehead puts it: "Every method of research creates its own applications: thus Analytical Geometry is a *different science* from Synthetic Geometry and both sciences are different from modern Projective Geometry. Many propositions are identical in all three sciences and the general subject matter, Space, is the same throughout. But it would be a serious mistake in the development of one of the three merely to take a list of the propositions as they occur in the others, and to endeavor to prove them by the methods of the one in hand. Some propositions could only be proved with great difficulty, some could hardly even be stated in the technical language or symbolism of the special branch."

the recognition by the Hindus of zero as a number, which marks an epoch in the history of thought. The failure of chronologists to recognize the number 0 complicates the calendar as was strikingly manifest in 1930. This year was chosen by the Italian Government and accepted by the civilized world for the celebration of the bimillennium of the poet Vergil, born in 70 B. C. The Society of Phi Beta Kappa in cooperation with the American Classical League sponsored memorial programs in schools and colleges throughout the United States. Likewise the bimillennium of Horace, born in 65 B. C. was celebrated in 1935. But since there is no zero year in the calendar, 1 B. C. and 1 A. D. denoting consecutive years, the time elapsed from the birth of each poet to his official bimillennium is not 2000 but 1999 years.

The whole of elementary algebra may be regarded as the direct generalization of arithmetic. Here we find the number system further extended to include negative numbers and imaginaries. The first is necessary to make subtraction universal and the second is required if negative numbers are to have square roots.[5]

As in arithmetic, so in algebra a large share of the time is devoted to extending the fundamental operations to the new kinds of numbers. The meanings of zero, fractional and negative exponents are assigned in an attempt to make the laws governing the use of integral exponents universal. As a by-product we obtain a firm foundation for the theory of radicals. Irrational and imaginary exponents are defined at a later stage and this theory leads among other things to an interpretation of the logarithms of negative numbers and brings together in one compact and elegant equation $e^{i\pi}+1=0$, due to Euler, "the five most interesting numbers in mathematics."

[5] A fascinating account of the growth of the idea of number may be found in *Number, the Language of Science*, by Dantzig, McMillan Co., publishers.

The roots of an equation in one unknown may be regarded as a generalization of the roots of numbers. Thus to find the square root of 3 is equivalent to solving the special quadratic $x^2-3=0$. This suggests the problem of finding the numbers which will satisfy the most general quadratic $ax^2+bx+c=0$. We then proceed naturally to the cubic and higher equations, the theory of which comprises the central theme of algebra.

What is the bearing of all this on the problem of teaching? The very recognition of the growth of mathematics through the process of generalization enables the teacher to present the subject as a steady development. This will require on the part of the teacher a comprehensive view of his subject, including some knowledge of its historical development. Fortunately ample sources are at hand in Smith's and Cajori's histories, Smith's *Source Book of Mathematics* and Klein's most stimulating and valuable book, *Elementary Mathematics from the Higher Standpoint*, Vol. I of which has recently been made available in English. If the method would merely provide us with better qualified teachers, it would be amply justified. But it is even more stimulating to the student, for he is thus taken behind the scenes to watch the subject unfold. And having his attention continually drawn to the method by which others have extended the concepts, processes and theorems, he may be encouraged to try his own hand at generalization. Whether he obtains new results or not is unimportant for the moment. His ability to recognize a new theorem as a generalization of an old one, makes the new one easier to understand and accept for one aspect of it is already familiar.

The reverse process is sometimes quite as satisfactory—to observe that a given general theorem has several special cases or corollaries, for it is sometimes as easy to prove a general theorem as a particular case. Frequently he may be able to see that a particular problem is only one

of a type. Thus, once he has learned the essence of the method of solving a quadratic by completing the square, he may apply the method to solving the general quadratic and thus derive a formula which yields the solution of *all quadratics*. Indeed every formula may be regarded as a statement in general terms of a relationship which has an infinite number of special cases, some one of which may have suggested the formula itself.

A quotation from the *Autocrat of the Breakfast Table* may not be out of place: "One of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economic and practical wisdom is an extension or variation of the following arithmetical formula: $2+2=4$. Every philosophical proposition has the more general character of the expression $a+b=c$. We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures."

Generalization in Geometry

But perhaps geometry furnishes the best field for the application of the method in high school teaching. Many illustrations will occur to you of the development of geometry through generalization. Thus we pass by successive stages from a square to a rhombus, to a general rectangle, the general parallelogram, the trapezoid, the general quadrangle. Congruent figures may be generalized into figures symmetrical as to a line and which cannot be made to coincide unless rotated out of their plane and turned over. We may also pass from congruent to equivalent figures which have the same size but not the same shape, or to similar figures which have the same shape but not the same size and finally to projective figures which have neither the same shape nor size. Again the idea of center as applied to a circle may be generalized to a point of symmetry, when a curve such as an ellipse, a hyperbola or a lemniscate has a center.

The transition from plane to solid geometry probably affords the best example of the method. Indeed, since the whole of plane geometry is merely a special case of the geometry of three dimensions, it would seem almost inevitable that the passage from the plane to space should be made by the process of generalization. Lines now go into planes, triangles into tetrahedrons, squares into cubes parallelograms into parallelopipeds, polygons into polyhedrons, circles into spheres. Plane geometry may also be generalized into spherical, the plane going into the sphere and lines into great circles, polygons into spherical polygons, etc. Now polygons symmetrical with respect to a plane or a point are no longer congruent.

Theorems may be generalized in several ways:

1. *Only the plane may be generalized*, thus giving the figures in question greater freedom. E.g., the theorem that two lines which are parallel to the same line are parallel to each other reads the same in space as in the plane. But the space theorem is more general for the lines now have more freedom. A similar remark applies to the theorem that the middle points of the sides of a quadrangle are the vertices of a parallelogram, whether the quadrangle is plane or skew. But the method cannot be used blindly. Thus the theorem that two lines either intersect or are parallel, while true in the plane is false in space.

2. Again several generalizations of the same proposition may be obtained by generalizing different parts of the figure to which it applies. Take for example the theorem that *two lines which are perpendicular to the same line are parallel*. Of the three lines involved we may generalize first the odd line, then the two parallel ones and we get two valid theorems:

a. *Two lines which are perpendicular to the same plane are parallel.*

b. *Two planes which are perpendicular to the same line are parallel.*

But if we merely give all three lines

the freedom of space, the theorem fails as it does if all three lines are generalized to planes, for the propositions then become: Two lines which are perpendicular to the same line are parallel and two planes which are perpendicular to the same plane are parallel.

3. Or we may combine both (1) and (2). For example the theorem about the medians of triangle may be extended by generalizing the triangle to a tetrahedron while keeping the medians as lines which however are now free to move in space. We have then: *Lines drawn from the vertices of a tetrahedron to the centroids of the opposite faces meet in a point which is three-fourths of the distance from each vertex to the opposite face.*

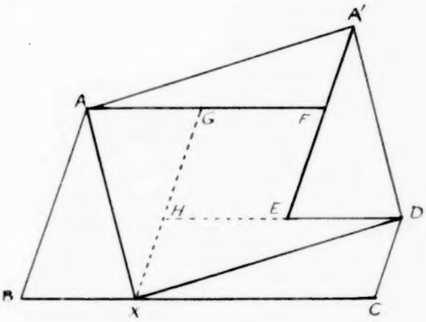
To be still more specific, let me exemplify the method by applying it to the famous theorem of Pythagoras. To begin with let us note the elements of the theorem which are candidates for generalization. Thus it is a theorem of the *plane*, which relates to the *areas* of *squares* constructed on the *sides* of a *right triangle*. Let us see now how these several items of plane, areas, squares, sides, right triangle can be generalized. We shall first consider generalization in the plane. Two generalizations are given by Euclid himself. Generalizing only the triangle we get what is now commonly called the law of cosines:

a. $a^2=b^2+c^2-2bc \cos A$ (Euclid II, 12, 13).

b. Next, the squares may be replaced by three similar polygons, the triangle remaining right angled (Euclid VI, 31).

c. A third generalization is due to Pappus of Alexandria, (c. 300). Let ABC be any triangle and describe any two parallelograms on AC and BC. Produce the sides which are parallel to the sides of the triangle to meet at D. Then the parallelogram under AB and a line which is equal and parallel to CD is equal to the sum of the first two parallelograms.[6]

Another generalization is given by DeMorgan (*Quarterly Journal of Mathematics*, I, p. 236) who makes the instructive remark that different demonstrations of a proposition usually point to different generalizations. Thus the extension by Pappus is based on Euclid's demonstration. That of DeMorgan is suggested by a dissection proof. His generalization may be stated as follows: Given the figure $ABCDEF$ formed by joining two parallelograms. In the side BC take $BX=ED$ and draw AX and DX. Then the parallelogram completed on the sides AX, DX is equal to the sum of the original parallelograms.



To prove, cut along AX and DX. Then slide triangle ABX, without turning, through the distance XD, when BX will fall on ED and BA on EA'. Similarly slide triangle DCX through the distance AX when XC will fall on AF and XD on AA'. The three pieces then fit together into a single parallelogram $AA'DX$, each of whose angles is the sum of a pair of angles of the two triangles. When the parallelograms are squares, we get a dissection proof of the Pythagorean theorem. Or we may state DeMorgan's generalization as follows:

d. *Given two triangles ABX and CDX with angles B and C supplementary. Then the parallelogram on the sides opposite the supplementary angles and inclined at the angle $(AXB+CXD)$ or $(BAX+CDX)$ is*

[6] Quoted in Heath's Euclid. See Smith, *History of Mathematics*, Vol. II, p. 289, or Palmer-Taylor-Farnum, *Plane Geometry, Revised*, p. 238, or Sykes-Comstock, *Plane Geometry*, p. 235, Ex. 21.

equal to the sum of the parallelograms under AB, CX $(=AF)$ and CD, $BX(=ED)$, inclined at either of the supplementary angles.

On drawing GX parallel to AB and producing DE to H, then cutting along GX and DH we have a dissection of the large parallelogram into a central parallelogram $EFGH$, bounded by four triangles, two congruent to each of the original triangles.[7]

Leaving the plane, we may ask whether instead of squares we may have cubes, or in general nth powers $a^n+b^n=c^n$. But Umpfenbach (*Crelle's Journal*, Vol. 26 (1843), p. 42) shows that $n=2$ is the only admissible value.

e. Gudermann (*Crelle's Journal*, Vol. 42 (1851), p. 280) gives a generalization to spherical right triangles but squares are replaced by spherical quadrangles which are both equilateral and equiangular.

We may also generalize the space and seek the analogue in space of three or more dimensions. First we observe that the form of statement of a theorem is an important factor in selecting an appropriate generalization. Thus the Pythagorean theorem may be stated in at least three ways, corresponding to the three accompanying figures.

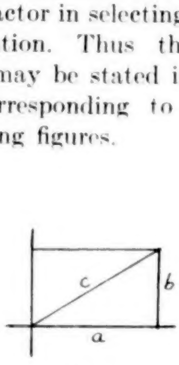


Fig. 1

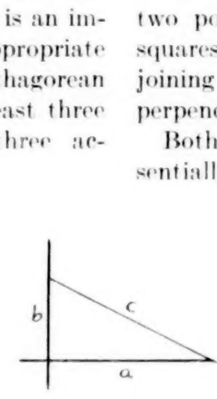


Fig. 2

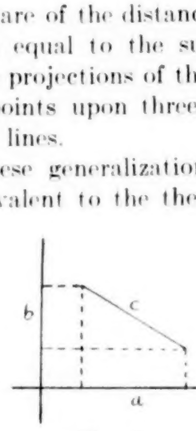


Fig. 3

1. The square of the distance of a point from the intersection of two perpendicular lines is equal to the sum of the squares of its distances from the lines.

2. The square of a line segment comprehended between two perpendicular lines is equal to the sum of the squares of the segments determined on the lines.

3. The square of a line segment is equal to the sum of the squares of its projections on any two perpendicular lines. We shall now give generalizations of each of these theorems. First, if we replace the two perpendicular lines in (1) by three mutually perpendicular planes, we have the space theorem:

1′. The square of the distance of a point in space from the intersection of three mutually perpendicular planes is equal to the sum of the squares of the distances of the point from the three planes.

Next in (3), giving the figure the freedom of space, when we have three mutually perpendicular lines, we have the generalization:

3′. The square of the distance between two points is equal to the sum of the squares of the projections of the segment joining the points upon three mutually perpendicular lines.

Both of these generalizations are essentially equivalent to the theorem that the square of the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its dimensions. The main difference is that in one case, a vertex of the parallelopiped is at the origin while in the other it is not.

To obtain a generalization of (2), let us inquire first how the idea of length of a segment can be extended. Now the line is a one-dimensional space and a segment is the simplest part of it that can be

[7] The reader will find it interesting to follow the successive steps with cardboard. (a) Cut out the figure ABCDEF. (b) Cut along AX and DX and fit the pieces into the parallelogram $AA'DX$. (c) Cut along GX and DH for the final dissection. The final figure when the original is formed from two squares is essentially that of Bhaskara, while (b) is then the figure generalized by DeMorgan. See Palmer-Taylor-Farnum, *Plane Geometry, Revised*, p. 220, Figs 4, 5; Or Sykes-Comstock, p. 238, Ex. 4.

bounded by elements of the next lower dimension, namely points. With the segment is associated a magnitude, namely length. What is the corresponding thing in the plane? The simplest figure in a plane that can be enclosed by lines is the triangle and associated with the triangle is the magnitude enclosed, the area. Similarly in space the simplest closed figure bounded by planes is the tetrahedron, the magnitude of the interior of which is the volume. As a general term to apply to these simplest figures in a space of a given number of dimensions, which are bounded by spaces of the next lower number of dimensions, *simplex* is used. And the magnitude of the portion of space enclosed by the simplex is called *content*.

Let us now return to proposition (2). Consider a trihedral angle each of whose face angles is right. A section of this by a plane is a triangle, which with the three right triangles cut from the faces of the trihedral angle forms a tetrahedron. The oblique triangle is the generalization of the hypotenuse in figure 2, while the three right triangles are the generalizations of the two perpendicular sides. To the contents, i.e., the lengths of the line segments in figure 2 correspond the contents or areas of the faces of the tetrahedron. We thus have the theorem (see accompanying figure):

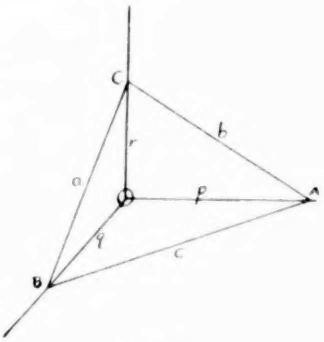


FIG. 2′

2′. *The square of the area of a triangle comprehended between three mutually perpendicular planes is equal to the sum of the squares of the areas of the right triangles determined in the planes.*

Proof. Let A, B, C be the vertices of a triangle cut from three mutually perpendicular planes meeting at O. Denote the lengths of the several lines involved by small letters as in the figure. We are to prove

$$(ABC)^2=(OAB)^2+(OAC)^2+(OBC)^2,$$

where (ABC) denotes the area of ABC, etc. Now from Hero's formula we have

$$(ABC)^2=[s(s-a)\ (s-b)\ (s-c)]$$
$$=[4a^2b^2-(a^2+b^2-c^2)^2]/16,$$

whence, replacing a, b, c by their equivalent values in p, q, r,

$$(ABC)^2=(p^2q^2+p^2r^2+q^2r^2)/4$$
$$=(OAB)^2+(OAC)^2+(OBC)^2. \qquad q.e.d.$$

Obviously, starting with any trihedral angle (vertex O) whose face angles are right angles, we can obtain a triangle ABC by merely cutting the faces with a plane. Let us ask now conversely whether it is possible to begin with any triangle and construct on it as base a triangular pyramid (apex O), each of whose angles is a right angle? Or to put it differently, suppose one found a spider in the corner of his book shelves—the corner in which he keeps his books on the teaching of mathematics—and he wished to pen up the spider by a perfectly fitting triangular lid—what sort of triangle would answer? Since a real solution is called for, we may say that a given triangle will do if and only if each of its angles is acute.

To prove the second part of this statement, we need only show that the angles of the triangle ABC must be acute. We have at once

$$a^2+b^2=p^2+q^2+2r^2=c^2+2r^2>c^2,$$

hence the angle C is acute. And so for the other two angles.

To prove the first part of the theorem, we shall solve the problem: Given three points A, B, C in space, to construct a point O such that OA, OB, OC shall be mutually perpendicular lines.

Now the locus in space of the vertex of a right angle whose hypotenuse is fixed is a sphere described on the hypotenuse

as diameter. Hence, O must lie on the three spheres of which AB, BC, CA are diameters. Aside from their common imaginary circle at infinity, these spheres will meet in two points, say O, O', which however will be

(a) real and distinct, if the triangle is acute

(b) real and coincident, if the triangle is right

(c) imaginary if the triangle is obtuse.

These statements are fairly obvious from a figure but they may be proved as follows.[8] The section of the spheres by the plane of ABC consists of three great circles drawn on the sides of the triangle as diameters. These circles meet in pairs at the feet of the altitudes.[9] Thus the altitudes are the diameters of the circles of intersection of the three spheres. If an angle of ABC is obtuse, the corresponding altitude lies inside the circle constructed on the opposite side as diameter. Hence in this case, the circle of intersection of two of the spheres lies wholly inside the third sphere, which proves (c). Again, if an angle of ABC is acute, the corresponding altitude lies partly inside and partly outside the circle drawn on the opposite side as diameter. Hence in case (a) the circle of intersection of any two of the spheres cuts the third sphere in two real and distinct points, O, O'. These points moreover are symmetrical with respect to the plane of the triangle, since all three spheres are. Hence if one of the points O or O' falls at the vertex of the right angle, as must happen in case (b), the other will also. The tetrahedron then collapses. A genuine solution of the spider problem thus requires an acute triangle.

Theorem 2′ may be extended to four dimensional space. *Thus if we have four mutually perpendicular lines meeting at O, and four points A, B, C, D, one on each of the lines, then the square of the volume of the tetrahedron $ABCD$ equals the sum of the squares of the volumes of the four tetrahedra having a common vertex at O.* Or

$$(ABCD)^2=(OABC)^2+(OABD)^2 +(OACD)^2+(OBCD)^2,$$

where the parentheses denote the volumes of the several tetrahedra.[10]

The converse is not in general true. That is, if we start with a general tetrahedron $ABCD$ in four dimensions, it is impossible to locate a point O such that the four lines OA, OB, OC, OD are mutually perpendicular. For the point O would have to lie on the six hyperspheres[11] drawn on the edges of the tetrahedron as diameters, and these in general do not have any point in common.

Another generalization of the Pythagorean theorem is obtained either by generalizing (3) in the same manner employed for (2) or by generalizing (2′). Thus, referring to Fig. 2′, it is clear that the right triangles are the projections of ABC on three mutually perpendicular planes. The projections of a general triangle however on three mutually perpendicular planes taken at random are not right triangles (see the spider problem above), but a theorem valid for all triangles is

3″. *The square of the area of any triangle in space is equal to the sum of the squares of the areas of the projections on any three mutually perpendicular planes.*[12]

The theorem is readily proved by means of the familiar relation between a figure and its (orthogonal) projection. Denote the three planes by x, y, z and let them meet at O. Let l be the line through O perpendicular to the plane of the triangle. Further denote the area of the triangle by T and the areas of its projections on the planes by T_x, T_y, T_z respectively. Then the angles of projection, say α, β, γ, are

[8] They also follow from the conditions $a^2+b^2>c^2$, etc.

[9] We use "altitudes" to mean the line segments from the vertices perpendicular to the opposite sides.

[10] The theorem is carried to n dimensions by L. Klug in the *Monatshefte für Mathematik und Physik*, Wien, 10 (1899) pp. 84–87.

[11] A hypersphere is the locus of a point in four dimensions which is at a constant distance from a fixed point.

[12] This theorem was known to Monge. The proof of 3″, which includes 2′ as a corollary, applies equally to any polygon.

the angles which l makes with the lines of intersection of the three planes. Therefore

$$T_x^2+T_y^2+T_z^2=T^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = T^2,$$

since the sum of the squares of the direction cosines of a line is unity.

We have presented thus far nine generalizations of the famous theorem—in the plane, on the sphere and in ordinary three-space. The last four of these, 1′, 2′, 3′, 3″ can be carried step by step to four, five and so on to n dimensions. The generalization of the last in n-dimensional space is Schläfli's theorem, which for four dimensions may be stated:

The square of the volume of a tetrahedron in four-dimensional space is equal to the sum of the squares of the volumes of the four tetrahedra obtained by projecting the first upon any four mutually perpendicular (three dimensional) spaces.

The foregoing instances not only exemplify the method of generalization in the elementary field but indicate the astonishing fertility of the Pythagorean theorem. One further example seems appropriate. In Euclidean differential geometry the element ds of the length of arc of a curve is given by the equation

$$ds^2=dx^2+dy^2+dz^2+ \ldots$$

which may be regarded as the generalization of theorems 1 and 1′ above. If the expression for the length of arc takes the more general form

$$ds^2=a\ dx^2+2h\ dx\ dy+b\ dy^2+c\ dz^2+ \ldots$$

where the coefficients depend on x, y, z etc., we are led to more general types of metrical geometry, of which Euclidean geometry is a limiting form, which include the non-Euclidean geometries of Bolyai-Lobachevski and Riemann, and which have found application in the theory of relativity.[13]

We shall conclude with the generalization of another familiar theorem, closely related to that of Pythagoras:

If a line l move so that

(a) *the sum of the squares of its intercepts on two rectangular axes, meeting at O, is a constant k^2, or*

(b) *the length of the segment comprehended between the axes is of constant length k,*

the locus of the center of the segment is a circle with center at O and radius k/2.[14]

In the plane, hypotheses (a) and (b) are equivalent but they lead to different generalizations in space. In both cases the line segment will generalize into a triangle whose vertices fall on three mutually perpendicular lines. (See Fig. 2′.) The center of the segment corresponds to the centroid (intersection of the medians) of the triangle. The intercepts of the line on the two axes will correspond to the intercepts of the plane of the triangle on the three axes. The space theorem on hypothesis (a) reads:

(a′) *If a plane, cutting three mutually perpendicular lines in points A, B, C, moves so that the sum of the squares of its intercepts on the lines is a constant k^2, the locus of the centroid of ABC is a sphere with center at the intersection of the lines and radius k/3.*

The theorem is easily proved analytically. Take the three lines as coordinate axes and denote the intercepts of the plane by p, q, r (Fig. 2′). Let x, y, z, denote the coordinates of the centroid P of the triangle. Then $p=3x$, $q=3y$, $r=3z$. Since by hypothesis $p^2+q^2+r^2=k^2$, we get at once the equation of the locus of P

$$x^2+y^2+z^2=k^2/9,$$

which represents a sphere with center at the origin and radius $k/3$.[15]

[13] For a development from this point of view, the reader may consult two papers by Pierpont in the *American Mathematical Monthly:* The Geometry of Riemann and Einstein, beginning in Vol. 30 (1923), p. 425.

[14] There are several other interesting loci connected with this line. Thus the locus of any point dividing the segment in the ratio $a:b$ is an ellipse, $x^2/a^2+y^2/b^2=1$. This is the principle utilized in a common form of ellipsograph. Again, the locus of the foot of the perpendicular upon the segment from O is a four-leaved rose. And the line itself envelops an astroid (hypocycloid of four cusps) in which the circle is inscribed.

[15] It can also be shown by pure geometry that the centroid P of the triangle lies on the

On hypothesis (b) the length of the segment corresponds to the area of the triangle, so the generalization would be:

(b′) *If a moving plane cuts three mutually perpendicular planes in a triangle of constant area k, the locus of the centroid of the triangle is a quartic surface.*

Now the area of the triangle (Theorem 2′) is

$$(ABC)^2=(p^2q^2+p^2r^2+q^2r^2)/4=k^2.$$

Hence the equation of the locus of the centroid is

$$81(x^2y^2+x^2z^2+y^2z^2)=4k^2,$$

which proves the theorem.

Both of these generalizations carry on step by step to n dimensions. Both lead to varieties of $n-1$ degrees of freedom, i.e., loci the coordinates of whose points are subject to but one restriction. The first is a generalized hypersphere—a locus of order two in each space—of radius k/n, while the second is of order $2(n-1)$. In four dimensions we should have:

If the vertices A, B, C, D of a tetrahedron move on four mutually perpendicular lines, (one vertex on each line), the locus of the centroid of the tetrahedron is a hypersphere (with center at the intersection of the lines and radius k/4) if the sum of the squares of the intercepts is constant; while if the volume is constant the locus is a sextic variety.

Finally, the method here outlined and illustrated is not intended for the fossilized teacher nor the subnormal student. It is aimed at the effectives rather than the defectives. But the teacher who really knows his subject, is imbued with the spirit of mathematics, and who has mastered the natural or inherent methods in teaching, need not be blown about by every wind of pedagogical doctrine; for like Prometheus of old he has defied the gods and stolen fire from heaven for the comfort of shivering mortals.

diagonal of the rectangular parallelopiped on OA, OB, OC as edges and at a distance from O equal to one-third the length of the diagonal. Under the hypothesis the end of the diagonal traces a sphere, hence P does also.

Program of the Second Summer Meeting

of the National Council of Teachers of Mathematics, Reed College, Portland, Oregon

Saturday, June 27, 10:00 A.M.

Work of Grades 7, 8, and 9

1. Mathematics: A Means of Expression (A Unit for grade 8). Edith Woolsey, Minneapolis, Minnesota
2. Meeting the Mathematical Needs of Non-College Preparatory Students in Washington. (Grade 9) Paul Wright, Davenport, Washington
3. Selected Topics. Edith L. Mossman, Berkeley, California
4. Round Table Discussion of Classroom Problems

Saturday, June 27, 2:00 P.M.

Arithmetic Section

1. Meaningful Teaching of the Mechanical Processes of Arithmetic. Dr. H. C. Christofferson, Miami University, Oxford, Ohio
2. Recent Changes in Point of View relating to The Teaching of Arithmetic. Dr. E. A. Bond, Washington State Normal School, Bellingham, Washington
3. Round Table Discussion on the Teaching of Arithmetic led by E. L. McDonnell of Seattle

Work of Grades 10, 11, and 12. 2:00 P.M.

1. Selling Mathematics. Anna M. Whitney, Yakima, Washington
2. Supplementary Teaching Materials in Geometry. H. W. Charlesworth, Denver, Colorado
3. Can We Avoid the Necessity for Reteaching Elementary Algebra in Advanced High School Courses. Elsie Parker Johnson, Oak Park, Illinois
4. How I Make Mathematics a Living, Useful Subject In and Out of the Classroom. Round Table Discussion.

Saturday June 27, 6:30 P.M.

Dinner

A Mathematical Genius and His Teachers. Dr. Eric Temple Bell, California Institute of Technology, Pasadena, California

Sunday—June 28

Outing to Bonneville Dam—A Trip on Oregon's Columbia River Highway

Monday, June 29, 9:00 A.M.

General Meeting

1. Placing College Freshman in Mathematics. Dr. W. E. Milne, Oregon State Agricultural College, Corvallis, Oregon
2. Mathematics: A Tool Subject or a System of Thought. Dr. F. L. Griffin, Reed College, Portland, Oregon.

Monday, June 29, 3:00 P.M.

Joint Conference—Department of Secondary Education in Cooperation with The National Council of Teachers of Mathematics

Central Topic: Trends in Mathematics Instruction in High Schools and Possible Use of new Curriculum Materials

1. Dr. H. C. Christofferson, Miama University, Oxford, Ohio.
2. Discussion

Headquarters: Reed College, Portland, Oregon

Registration, Meetings and Mathematics Exhibit: Elliott Hall, Reed College

Rooms and Meals in Reed College Dormitories. Most of the rooms are suites, i.e. a sitting room and bedroom with two single beds. These accommodations for June 26, 27 and 28 with breakfast on Saturday, Sunday and Monday, lunch and dinner on Saturday and supper on Sunday will cost $5.00 per person for two to a suite or $6.00 for one. Single meals at reasonable rates. Accommodations may be had for the entire time of the N.E.A.Meetings.

Reservations should be sent to Miss Lesta Hoel, Grant High School, Portland, Oregon not later than June 22.

For Hotel rates and reservations address all requests to E. B. McNaughton, Chairman N.E.A. Housing Committee, 631 N.E. Clackamas Street, Portland.

The National Council of Teachers of Mathematics

Report of the Treasurer for the Year, February 9, 1935 to February 9, 1936

Balance on Hand at beginning of year:			
Union National Bank of Macomb, Ill.	$1118.85		
Savings Bank Deposit	630.14		
New York Telephone Bond, 4½%, 1939	984.53		
Commonwealth Edison Bond, 5%, 1953	933.99		
Accrued Book Value of Bonds	5.60		$3673.11
Receipts for the year:			
W. D. Reeve, Yearbooks	$ 290.50		
Mathematics Teacher	1008.57		
		$1299.07	
Bureau of Publications, Yearbooks		750.94	
($1550.55 credit towards 11th Yr. Bk.)			
Interest on Bonds		94.00	
Interest on Savings		11.65	2155.66
			$5828.77
Expenditures for the year:			
16th Annual meeting, Atlantic City:			
Directors' Expenses	$ 613.33		
Speakers' Expenses	257.00		
Banquet Expense	39.90		
Local Committee	22.82		
		933.05	
First Summer Meeting Denver:			
Directors' Expense	121.98		
Speakers' Expense	50.00		
Local Committee	13.86		
		185.84	
17th Annual Meeting, St. Louis:			
Directors' Expenses	532.97		
Speakers' Expenses	80.00		
Local Committee	47.59		
		660.56	
Office of the President		19.32	
Policy Committee		1.63	
Office of the Secretary-Treasurer:			
Stationery	34.43		
Postage, Supplies, etc.	118.85		
Tax on out-of-town checks	.15		
Secretary Service	350.00	503.43	2303.83
			$3524.94

Balance Sheet, February 9, 1936

Assets:		*For 1935*
Commercial Bank Deposit	$ 714.14	($1118.85)
New York Telephone Bond	987.92	(984.53)
Commonwealth Edison Bond	936.20	(933.99)
Savings Bank Deposit	886.68	(630.14)
	$3524.94	($3667.51)
Liabilities:		
None		
Net Worth	$3524.94	($3667.51)

(Signed) Edwin W. Schreiber, *Treasurer*

The above account has been audited and found correct

(Signed) W. S. Schlauch, *Auditor*

EDITORIALS

Importance of Training in Subject-Matter

THOSE who believe that the importance of subject-matter training in mathematics and related fields as a part of the equipment of a teacher should read a recent report of the American Chemical Society's Committee on high school teaching of chemistry which was adopted unanimously at a recent meeting of the Council of the Society in Kansas City, Mo. The report charges that "the high school students now entering our universities and who have entered within the last ten years are much inferior in preparation in mathematics and other fundamental and basic courses to similar students of a generation ago, and the situation is tending, if possible, toward a worse condition."

The report goes on to say that a large number of teachers of science in the high schools of a number of States are teaching without having had any previous training in the subject, and a still larger number of teachers have had but elementary training in the subjects they teach.

The report further says that in December 1932 the certificate system was in use in twenty-seven states; the system grants an unrestricted teaching certificate, which permits the holder "to teach any and all academic subjects at his or her own volition, or when required to do so by the principal or superintendent in question."

The Committee states that "In none of the states granting 'unrestricted' certification is there a specific chemistry requirement, nor apparently, a specific 'science' requirement."

Among other things the report recommends "that the American Chemical Society pronounce itself as opposed to 'unrestricted certification' of high school teachers, by means of which teachers are at present teaching subject-matter courses in which they themselves have had no previous training, and in favor of 'restricted certification' whereby only those persons who have had a specified amount of collegiate training in a particular subject-matter field will be permitted to teach in that subject-matter field in the high schools."

This situation can be matched in other subject-matter fields than chemistry and it is high time that the professors of subject-matter fields in the colleges and in the universities combined and cooperated with the teachers of subject-matter in the high school to make it impossible for people with unsatisfactory subject-matter training to be permitted to teach. Moreover, it is time for the Teachers Colleges to decrease the amount of so called general education courses where they are now obviously too numerous and increase the number of subject-matter courses so as to give teachers a satisfactory knowledge of the subject they are expected to teach.

Why should not the National Commission on *The Place of Mathematics in the Secondary Schools* solicit the cooperation of the science teachers in this country in raising the standards for high school teachers along subject-matter lines.

Is Mathematics Losing Ground?

THE COMMENTS above naturally lead one to ask "Is mathematics losing ground in the schools?" In so far as the long view is concerned it is safe to conclude that the place of mathematics is secure. Without a knowledge of mathematics civilization would crumble and go down. On the other hand, it is clear that in many schools

mathematics is no longer required if it is offered at all. Much of the trouble in which we find ourselves can be laid to the formal organization and teaching of the subject, but also to the fact that some of our friends among the College and University professors have abdicated in favor of the general educators. However, a great deal of our trouble comes from the fact that many of our people do not think or are guilty of a kind of loose thinking which leads them to accept as final paneceas that need further substantiation before they can be called reliable and scientific solutions to the problems of education.

In the last thirty years we have greeted one panacea after another with frantic enthusiasm only later to learn that we have been duped again. Witness the communities that are now said to be planning to introduce the so-called "Manchester Plan" of teaching arithmetic which is supposed to show that by virtually eliminating the teaching of arithmetic one can get even better results in arithmetic than heretofore.

Moreover, school administrators are dropping algebra from the schools as a required subject without adequate consideration of the results. According to *The Phi Delta Kappan* for March 1936 (p. 230) "St. Louis high school pupils will no longer be required to take elementary algebra, but may, if they wish, substitute a course in the history of Missouri, a social subject. This change, according to *Missouri schools*, is the beginning of a revision of studies intended to eliminate as far as possible pupils' failures." This is something for the St. Louis teachers of mathematics to think about, but it is also an indication of what is going to happen in other places unless teachers of mathematics are properly alert.

Mathematics is losing ground in some places and doubtless will continue to lose ground in others, but in the long run we shall come back to a sensible view with respect to the place of this important subject in the education of our youth. Those who are interested in such a return to sanity should read the Eleventh Yearbook of the National Council of Teachers of Mathematics on "The Place of Mathematics in Modern Education."

In Defense of a Subject Curriculum

It is gratifying to find such a sane view of the importance of subject matter as is expressed by the following editorial from "Educational Method" for April 1936, by F. M. Underwood, Assistant Superintendent of the St. Louis Public Schools:

"It is natural for curriculum makers to want something new. Hence we hear a great deal, from workers in this field, of an activity curriculum, a project curriculum, or a large unit curriculum. The above terms are practically synonymous, though there may be a difference in emphasis, the first emphasizing physical activity, the second one the pupil's purpose, and the third the size of the teaching unit—especially the correlation and integration of the elements involved. The 'project' devotees would say the pupil's purpose determines the size of the activity and brings about correlation and integration, and the 'activity' people would have no quarrel with either, so long as there was evidence of enough pupil activity. The advocates of each of these points of view toward the curriculum oppose the subject curriculum.

"Let us see how what they have to substitute works out. There has for several years been great activity in working out curriculum 'units.' These constitute particular topics, activities, problems, projects. They all relate to 'experience' and 'life' in some way. However, each is worked out originally, independently, and individually. The elements within the 'unit' are all related to each other and to the pupil, but the various units are not related to each other. Hundreds of these 'units' have been worked out. It would seem perhaps thousands would have to be

worked out before the elementary school curriculum would be complete. When this is all done, we are offered thousands of individual 'units' of work as a substitute for the subject curriculum. It is quite evident that this great mass of unsatisfactorily related 'units' would be unacceptable as a curriculum.

"What is a 'subject' as far as the curriculum is concerned? It is merely one phase or division of 'phenomena,' 'life,' or 'experience,' set apart for systematized investigation and study. Man has built up his scientific knowledge by concentrating his attention on one field of knowledge at a time, so that relationships and principles in this limited field could be discovered. So it has been as to physics, chemistry, biology, psychology, sociology, and other sciences. So these sciences were built up. Is it not reasonable to suppose that a student of today could most effectively discover relationships and build up his concepts in the same way? Knowledge, to be useful, must be organized, classified, and systematized. Can this be done apart from the study of the various sciences as such? What are our curriculum experts going to do with all these thousands of 'units' which they have built up *psychologically* but not *logically?* After they have them, they must organize, classify, and systematize if they are to make the knowledge serviceable and satisfying. When they do this, they will establish *subjects* again. They may not establish the same subject we now have, but they will have subjects just the same. It is clearly evident that a large mass of unrelated 'units' will be wholly unacceptable. The Herbartian theory of teaching was a great advance in that the relationships of things in the outside world to one another were systematically and thoroughly developed. What was lacking was that these things were inadequately related to the pupil in a vital way. This was accomplished through the 'problem method' as developed by Dewey and McMurry and the 'project method' as developed by Kilpatrick. But why abandon all the gains made toward establishing adequate relationships and order among things and events in the outside world just to establish the child's relationship to individual situations? Let us keep the problem and project concepts and practices in teaching, but let us also keep a systematic organization of knowledge on a subject basis, for effective service in solving the problems of life.

"Until a new set of methods is proposed, and until it is evident that the new subjects possess superiority to the present list, it might be well to hold on to our present list. It may be well to remember, too, that the biggest problem in education is a *teaching* problem, not a *curriculum* problem. Even with the *subject curriculum*. by the introduction of the 'project,' 'problem,' 'activity,' and 'large unit' procedures in teaching and learning, within the subjects, together with the introduction of certain correlating and integrating activities in addition to the work in the regular subjects, all essential purposes and ends of the 'project,' 'problem,' 'activity,' and 'large unit' theories of teaching can be realized without the necessity of an entirely new curriculum. What we need is reform in teaching procedures rather than a new type of curriculum. We need reform in the use of the textbook, not the elimination of the textbook and the abandonment of the subject curriculum.

"Modern teaching, with a subject curriculum, seems to give greatest promise of meeting our needs satisfactorily."

Teachers of mathematics should give wide publicity to statements like the above that come from people in responsible positions in order to offset the influence of those who would try to drive subjects from the course.

W. D. R.

NEWS NOTES

The Tulsa (Oklahoma) Mathematics Council now has a charter from the National Council, dated December 14, 1935. It carries the signature of the national secretary, Edwin W. Schreiber, and the corporate seal of the organization. We have it framed. It was formally presented to our president, Miss Kate Cannon, by our secretary, Miss Narcissa Bond, at the regular quarterly meeting in Central High School, April 16, 1936. We feel that this formal evidence of national recognition marks the successful completion of our efforts toward local unity.

Oklahoma has a state organization whose president, John A. Venable is a Tulsa teacher. The purpose of the state organization is to promote and improve the teaching of mathematics in Oklahoma.

Oklahoma ranked seventeenth in national council memberships in April, 1935, according to Mr. Schreiber's report in the March number of *The Mathematics Teacher*. We are eager to see this year's figures. We believe that our local and state campaigns have added considerably to our total membership. The Tulsa Council urges other cities to affiliate with the National Council. This is a very effective way to indicate professional growth and to promote the cause of secondary mathematics.

At our recent meeting, we were honored in having our assistant superintendent, Eli Foster, address the Council on the subject, "The Place of Mathematics in the New Education." Mr. Foster accorded the teaching of mathematics a place of honor in the high school curriculum. This attitude on the part of administrators is necessary to our future existence. I am sorry to say, it is too rarely found out here in the wide open spaces.

L. W. LAVENGOOD

The Twenty-second Annual Meeting of the Kansas Section of the Mathematical Association of America and the thirty-second annual meeting of the Kansas Association of Mathematics Teachers was held at the Topeka High School on March 14, 1936.

FORENOON SESSION

10:00 A.M.—Room 213

Joint Session—K. A. M. T. and M. A. A.

W. T. Stratton, Kansas State College, *presiding*

1. "The Slide Rule in the Solution of Cubic Equations"
   L. E. Curfman, Kansas State Teachers College, Pittsburg
2. "Field Work in Mathematics for the High School"
   Esther Nicklin, Central Junior High School, Kansas City
3. "The Number 'e' "
   U. G. Mitchell, University of Kansas, Lawrence

Luncheon—High School Cafeteria
Edna E. Austin, Topeka, *in charge*

AFTERNOON SESSION

1:15 P.M.—Room 213

Joint Session—K. A. M. T. and M. A. A.

1. Business Session:
   Report of Secretary-Treasurer, Dessie Myers, Kansas City
   Report of Nominating Committee, J. J. Wheeler, O. J. Peterson, and Mary Kelly
   Election of Officers

AFTERNOON SESSION

Room 213
K. A. M. T.

W. H. Hill, Kansas State Teachers College, Pittsburg, chairman, *presiding*

1. "Interesting Facts from the History of Mathematics"
   Emma Hyde, Kansas State College, Manhattan
2. "Revising the Curriculum for Ninth Grade Mathematics"
   Celia H. Canine, Wichita
3. "A Present Day Algebra for Tomorrow"
   Daniel B. Pease, Douglass
4. "The Use of Instruments in Teaching High School Mathematics"
   E. A. Beito, University of Wichita

AFTERNOON SESSION

Room 236
M. A. A.

W. T. Stratton, Kansas State College, chairman, *presiding*

1. "Line Configurations and Group Properties"
   W. G. Warnock, Fort Hays State College
2. "A Certain Paradoxical Property of Conditionally Convergent Series"
   Charles B. Tucker, Kansas State Teachers College, Emporia

3. "A Depleted Fourier Series"
C. F. Lewis, Kansas State College, Manhattan
4. "On the Waring Problem"
(Miss) M. G. Humphreys, Mount St. Scholastica College, Atchison
5. "On a Classification of Integral Functions"
M. T. Bird, Southwestern College, Winfield

Officers
of the
Kansas Section
of the
Mathematical Association
of America

Chairman, W. T. Stratton, Kansas State College

Vice-Chairman, R. G. Smith, Kansas State Teachers College, Pittsburg

Sec.-Treas., Lucy T. Dougherty, Kansas City Junior College

Officers
of the
Kansas Association of
Mathematics Teachers

President, W. H. Hill, Pittsburg

Vice-President, Ruthetta Owsley, Atchison; Minnie Stewart, Topeka; W. B. Wise, Dighton; J. G. Taylor, Ellsworth; Alden Eberly, Scott City; Marie Stewart, Hutchinson; W. H. Hill, Pittsburg

Sec.-Treas., Dessie Myers, Kansas City

Editor of the *Bulletin*, Ina E. Holroyd, Kansas State College

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The Exhibit of the Women's Mathematics Club of Chicago and Vicinity held in the Club Woman's Bureau at Mandel Brothers was most worthwhile; the Club spent weeks planning for its material; when completed, the exhibit contained a complete set of modern texts for the teaching of mathematics, examples of all the modern tools and instruments used in secondary schools, placards and signs showing the development of mathematical terms and the excellent collection of very old mathematics text books loaned by the Amour Institute. The Mathematics Exhibit was opened with a luncheon attended by 127 men and women teachers all of whom must have passed around word of the event, judging by the number of teachers who later came in to see and study the Exhibit. Among the visitors were men and women of high position in the teaching of mathematics, including Dr. Lewis Karpinski of the University of Michigan, an outstanding authority in the History of Mathematics. Through the good offices of the Mathematics Club most of the new text books which were on display have been sent from here to the Chicago Public Library for permanent display in the Teacher's Room of the Library.

Professor W. D. Reeve of Teachers College, Columbia University was the guest and speaker at the Annual Banquet of the Cleveland Mathematics Club which was held at the chamber of Commerce Club in Cleveland on March 24, 1936.

On March 20th the Men's Mathematics Club of Chicago was entertained by two speakers The first was C. E. Stryker, chief engineer of the Fansteel Metallurgical Corporation. He gave an interesting and informative talk on the use of rare metals.

The second was Dr. Louis C. Karpinski of the University of Michigan. He gave an illustrated lecture on the History of Algebra. His unique method of entertaining while he described the slides was fully appreciated.

The colored photographs of the Tree of Knowledge which can now be had from the Rosenwald Museum of Jackson Park for 25¢ postpaid was advertised.

The third meeting of the Range Mathematics and Science Clubs for the current school year was held at Hibbing, Minn. on February 28 at 6:00 P.M. This was a joint dinner meeting for the Range Mathematics and Science teachers attended by forty mathematics and thirty-two science instructors.

The program for the joint session of the two groups consisted of: an address of welcome by Dr. P. B. Jacobson, Supervisor of Secondary Education, Hibbing Senior High School; and an address on the topic "The Application of Mathematics to some of the ImportantProblems of Today" by Mr. L. A. Rossman, Editor, Grand Rapids Herald Review, Grand Rapids, Minnesota.

In a talk to the mathematics teachers Miss Leone Schuster, Hibbing Senior High School, discussed the topic "Geometry and Architecture."

Mr. W. G. Crawford, Bureau of Mining Research, Chisholm, continued the discussion on "The Chemistry of the Mesabe Ores." Mr. Odin Sundness, Snyder Mining Company, gave a talk on a similar topic at the Virginia meeting of the organization on December 5th.

The meeting was arranged by a committee consisting of: Miss Leone Schuster, Miss Louise Gellerman, Mr. M. E. Gilenas of the Hibbing Senior High School; Miss Margaretta Reynolds,

Hibbing Junior High School; Dr. L. M. Becker, Hibbing Junior College. Mr. Sander Lawrence, Hibbing, and Mr. W. A. Porter, Chisholm, made the necessary arrangements for the meeting of the science instructors.—H. G. TIEDEMANN.

The Women's Mathematics Club of Chicago held a luncheon meeting on April 18th at the Piccadilly. There was a report by the Misses Gladys Chrisman, Clara Haertel and Ionia J. Rehm on the topic "A Mathematics Field Course in Germany with Professor Reeve in 1935."

Teachers of mathematics who happen to be in New York City are urged to see the permanent mathematics exhibit at Teachers College, Columbia University under the direction of Professor W. D. Reeve. It is the purpose of the exhibit to show the relation of mathematics to the other great fields of knowledge. Teachers may not only find a great deal of useful material in the exhibit, but will also be able to get some ideas of how such an exhibit may be shown to advantage.

By the courtesy of the Journal of Educational Research we are permitted to reprint the following:

To Eliminate College Entrance Examinations

"To give opportunity for secondary schools to develop more effective programs of education, some two hundred and eighty of the American colleges and universities have entered into an agreement with thirty outstanding secondary schools, selected by the Progressive Education Association, to admit their graduates without examination, and without prejudice with respect to their previous program of studies, until June 1941. The only requirement is evidence that the candidate has the requisite general intelligence to carry on college work creditably, that he has well defined serious interests and purposes, and that he has demonstrated ability to work successfully in one or more fields in which the college offers instruction.

"The success of the experiment from the academic point of view is practically assured. The pupils in these thirty schools are a highly selected group, they have been exposed to stimulating instruction, and they will be carefully guided into the appropriate colleges. Such pupils would probably make an impressive record in college no matter what they had studied in high school. From the research point of view, therefore, the success of the experiment rests upon the ability of the thirty schools to offer convincing, scientific evidence that the newer programs they have planned in terms of what they know about the needs and interests of their own pupils are more likely to bring about desired changes in pupil behavior than the program hithertofore imposed by college requirements. It also rests upon the ability of the schools to furnish more helpful, significant information about their graduates for admission and guidance purposes than the college can obtain from entrance examinations and from the entrance data now commonly used.

"Both of these criteria imply a comprehensive, scientific evaluation program, but one evolved by the schools to suit their own purposes, not one imposed by an examining committee. The responsibility for recording, interpreting, and reporting changes in pupil behavior has, therefore, been left entirely with each of the thirty schools. They are assisted, however, by a Committee on Records and Reports under the chairmanship of Eugene Randolph Smith, Head Master of the Beaver Country Day School. This is a policy forming committee; the active work of visiting the schools and assisting them in devising tests, record forms, and reports appropriate to their needs is carried on by a full-time professional Evaluation Staff directed by Ralph W. Tyler, of Ohio State University.

"Professor Tyler has initiated a four-point program of evaluation. First, the schools state their major objectives in terms of the important changes in the pupils' everyday behavior which they are trying to bring about. Second, they list the situations in the normal processes of school life in which progress toward the desired types of behavior might be observed. Third, they devise a reliable, economical, and meaningful record of pupil behavior in these situations. Fourth, they summarize, interpret, and report their findings to college admissions and guidance officers and to other persons who can use the information. The conclusions drawn from the record must be scientifically valid and accurate.

"During the first year of the work of the Evaluation Staff, six committees were set up to devise new instruments for measuring some of the more intangible outcomes of the progressive secondary education which were of common interest to the majority of the schools: sensitivity to significant problems, attitudes, and interests, application of principles, interpretation of data, work habits and skills, and the many types of behavior which at present can be evaluated only through anecdotal records. Interested teachers from most of the schools participated, and about twenty new instruments of evaluation were tried out and put into operation in their schools. Many more are in the process of development, and the existing instruments of

evaluation in these areas were widely studied and applied.

"The work of these committees will continue until the instruments in the process of development have been completed. Then other committees will be formed, directed toward the evaluation of progress toward other common objectives of the thirty schools for which no satisfactory means of evaluation at present exists. In addition to the work of the committees, individual teachers who have promising ideas for new instruments of evaluation will be released from their teaching duties for periods of two or three weeks and brought to the headuqarters of the Evaluation Staff in Columbus, Ohio, where they may devote their entire time to this work, having at their disposal a complete testing laboratory and technical assistance. Members of the Evaluation Staff will also continue to visit schools to help organize within each school a comprehensive evaluation program and to exchange ideas from school to school.

"The idea of recording progress toward all the major objectives in which the schools are interested has had a stimulating effect upon the thirty schools. Far from hindering the work of teaching, this sort of evaluation is commonly said to have stimulated more fundamental thinking upon the effectiveness of teaching procedures than any other influence operating within the experiment. Heads of schools have come to realize the importance of an evaluation program not only in demonstrating the effectiveness of instruction but in heightening its effectiveness. It is hoped that before the conclusion of the experiment this evaluation program will yield incontrovertible evidence of the success of the newer programs of secondary education, and will furnish so much more significant and helpful information about candidates for admission to college that there will be no desire to return to college entrance requirements and examinations."

This information was provided by R. W. Tyler and P. B. Diederich.

An ingenious method for determining the day of the week corresponding to any given date, presented in the February, 1936 issue of *The Mathematics Teachers* seems to work rapidly, although the accompanying formula seems unnecessarily complicated. Unfortunately the article tends to create some false impressions. Any date is computed by the Gregorian calendar, then adjusted to the Julian calendar. Failure to carefully note this fact, coupled with other statements in the article, might lead to the improper conclusion that years such as 100, 900, or 1000 were not leap years.

The explanation of the method asserts that the year one began on Monday, as contrasted with the Encyclopaedia Britannica article on the calendar, which states that the Christian Era began with Saturday. Likewise there is a discrepancy in that the article in the *Teacher* gives the rule that years divisible by 4000 are not leap years; other authorities state that this has been proposed, but is not yet agreed upon.

H. E. Licks in his Recreations in Mathematics (D. Van Nostrand Company) gives on page 120 the formulas
$S=Y+D+(Y-1)/4-(Y-1)/100+(Y-1)/400$
for the Gregorian calendar and $S=Y+D+(Y-1)/4-2$ for the Julian calendar. Y represents any year and D the day of the year. Fractions in the divisions are ignored. If S be divided by 7, the remainder gives the day of the week, 0 indicating Saturday; 1 Sunday, etc. For example: October 12, 1492. Here $Y=1492$ and $D=286$, from which $S=2148$, which divided by 7 yields remainder 6. America was discovered on Friday.

The use of two formulas allows consideration of the fact that all countries did not adopt the Gregorian calendar at the same time, Great Britain not making the change until 1752. Since all persons are familiar with the number of days in each month, the necessity for computing the day of the year would probably cause no more difficulty than the memorization of index numbers for certain centuries and for the twelve months.—Cecil B. Read, University of Wichita, Wichita, Kan.

Professor A. A. Bennett of Brown University will give a series of lectures throughout the academic year 1936–37 on Monday evenings from 7:30–9:10 at Teachers College, Columbia University. The first semester he will give a course on "Selected Topics in Numerical Computation for Secondary School Teachers." This course will deal with modern developments in the theory of applied arithmetic within the secondary school field, with special reference to approximate computation, mathematics of finance and elementary statistics. The second semester will be devoted to "Some Modern Aspects of Euclidean Geometry" where the secondary teacher of mathematics will have a chance to explore the algebraic setting of Euclidean geometry.

Announcing a Radio Course in the Teaching of Mathematics

In the first semester of the school year 1936–37, the School of Education of the University

of Michigan will offer for experimental trial, Education D135, the Teaching of Mathematics. The course will be administered by the University Extension Division and will be offered by a combination of radio and correspondence instruction. The instructor is Dr. Raleigh Schorling who will be assisted by teachers in the laboratory school, professors of mathematics in the department of the College, and guest teachers and supervisors. The course will include round-table discussions, lectures, and demonstrations.

Procedure. (1) Early in the fall the student should enroll with the Extension Division for course D135. (2) Each Saturday morning for 18 weeks the student listens to a radio program broadcast over WJR at 9:00 A.M., E.S.T. (3) The student reads the unit of work appearing in the monograph which corresponds to the radio broadcast. (4) The student does the work required on the Guide Sheet and mails one Guide Sheet each week to the Extension Division. For example, the first broadcast is based on Unit I in the monograph to be described presently which requires the student to do the work on Guide Sheet No. 1 the first week and mail it in to the Extension Division. This procedure continues for 18 weeks, the duration of the course.

The Monograph. The monograph is sent to the student as soon as his registration is complete. It is the main part of the course and will carry the student to the important parts of the literature relating to the teaching of mathematics for grades seven to twelve, inclusive.

The monograph will include five types of materials: (1) Articles specifically written for the course; (2) extensive reprints; (3) digests of important references; (4) summaries of principles relating to the various topics in so far as they are now known as the result of experimentation, investigation, and competent opinion; and (5) illustration of principles.

Guide Sheet. The work of each week, as has been stated, will be covered by a guide sheet. The student is expected to do the tasks which carry him into the literature and to mail a guide sheet to the Extension Division each week. The Guide Sheet will require the student to write a brief abstract limited to 100 words of the corresponding radio broadcast, listing the important ideas presented in the broadcast.

Radio Talks. The radio talks for each week will introduce the student to the new unit. The series includes considerable variety, both as regards subject matter and methods of presentation. The procedures employed are demonstration lessons, conferences, and lectures.

Tests. The student will be required to take from six to ten tests during the semester covering both subject matter and pedagogy.

Credit. The student who does the work successfully will be given two hours of undergraduate credit. In order to use this credit in the School of Education of the University of Michigan, the student must be of junior standing when he enters the University and fulfill the requirements of either a minor or major in mathematics. Students with less than junior standing and planning to use the course in some other institution are advised to address inquiries concerning the credit to the institution concerned.

Fees. The Extension Division has not officially fixed the tuition fee but it is safe to guess that the cost will be in the neighborhood of $10 plus a small sum to cover the monographs, guide sheets, and the like.

NEW BOOKS

A Short History of Science. By W. T. Sedgwick and H. W. Tyler, Macmillan. 1927. XV+474. Price, $3.

This book is the outgrowth of a lecture course given by the authors for several years to undergraduate classes at Massachusetts Institute of Technology. The book furnishes a broad general perspective of the evolution of science. There are excellent treatises on the history of particular sciences as well. The book should give the reader a concise account of the origin of that scientific knowledge and method which have recently come to have such an important part in shaping modern conditions and in directing life activities.

Teachers of mathematics will find much of interest and help to them in this book.

Practical Mathematics. By N. J. Tennes. Macmillan, 1936. XI+400. Price, $1.20.

This book is intended as a text for high-school freshmen. The emphasis is on the practical side. The book presupposes no preparation beyond ordinary arithmetic and contains very little of the standard algebraic or geometric material of the conventional type.

Obviously the book is intended to serve the needs of those pupils who now study no mathematics at all or who have great difficulty with the traditional type of text. It remains to be seen whether these newer books somewhat along the general mathematics type but with the practical type of mathematics stressed will solve the problem of the ninth grade. There are those who think that even for the slow-moving pupil the cultural value of mathematics should not be overlooked.

New topics like installment buying are given careful treatment in this book. It will be of interest to teachers of arithmetic who need such assistance.

Brief Analytic Geometry. By Thomas E. Mason and Clifton T. Hagard. Ginn, 1935. XI+196. Price, $2.00.

This book is an abridgment of the authors' *Analytic Geometry* so as to adapt it to a class schedule of from fifty to sixty lessons. With the present tendency in the colleges to reduce the course in trigonometry and analytic geometry this book seems to be timely.

The chapter on the straight line has been put ahead of the chapter on the general problem of equation and locus.

Many new exercises have been included and all of the exercises are designed to develop methods of investigation rather than to teach specific properties of certain curves and surfaces. There are enough exercises in the book to enable the teacher to make a wise selection.

The book has an attractive appearance and should meet with approval.

Advanced Algebra. By P. H. Graham and F. W. John. Prentice-Hall, 1936. XIV 262. Price, $1.85.

This book is a revision of an earlier edition by the same authors. While it contains the main features of the original text it has minor changes which are intended to make the book more teachable. This is done by replacing the difficult problems by simpler ones which illustrate the fundamental principles as well as the harder exercises.

The work on variation has been enlarged so as to give more explanation and a better understanding of the principles.

Considering the importance of a good understanding of algebra in the mathematical scheme this book ought to be very helpful to those who wish to have a good background for the work later on.

Business Mathematics. By I. L. Miller. D. Van Nostrand, 1935. X+376. Price, $3.50.

This book is an outgrowth of a course in Business Mathematics which the author and his colleagues have worked out at South Dakota State College in the past dozen years. There is no question but that a book of this kind is timely.

Much of the material of this book might well be presented to those seniors in the high school who may otherwise go out into life situations ill prepared to understand some of the most important arithmetical and business operations. At present there is little place for such materia in our ordinary curricula.

With the country as a whole suffering from the effects of an ignorance of such matters as are treated in this text, this book should prove most helpful in schools where teachers see the need for its use.

Algebra—A Way of Thinking. By U. G. Mitchell and Helen M. Walker. Harcourt, Brace and Co., 1936. XVI+400. Price, $1.24.

This new and interesting book will be a great contribution to the field of mathematics if it can stimulate the pupil to appreciate Algebra as a method of thinking. At the present time most of the pupils who study algebra have not advanced far beyond the mechanical aspects of the subject and many pupils who have studied algebra for a year and a half do not really understand what it is all about.

This text is intended to differ from others in the following respects:

1. In the method used to develop each topic so that an idea precedes the symbolism used to express that idea.

2. In the early introduction of the formula and its use to facilitate the teaching of all other work.

3. In a new kind of treatment of the function concept to promote a clear understanding of variables and the numbers they represent.

4. In the introduction of the mathematical graph before the statistical graph.

5. In a treatment of the statistical method which is well within the understanding of beginning high school pupils.

6. In the provision for individual differences. The newer features of this book are sure to appeal to teachers of mathematics and with the exception of the retention of a few difficult items such as quadratic equations to be solved by formula the book is unusually progressive.

This book like all of the other recent books on algebra errs by retaining some of the obsolete processes in factoring and in polynominals. Once these operations got in our American texts it has been almost impossible to get rid of them. Now that such processes are no longer included in extra-mural examinations it seems wise for us to omit them from textbooks and courses of study.

In spite of a few minor points of this type the book has many redeeming features and should certainly be carefully studied by all teachers of algebra.

Arithmetic for Business Training. By Alexander Fichandler, Louis Slatkin, and Murray Melzak. Globe Book Co., 1936. V+163. Price, $1.

This book is an attempt to place in a separate text (the authors claim for the first time so far as they are aware) arithmetical problems designed specifically to correlate with the topics generally prescribed in the course of study in Elementary Business Training.

Each topic is preceded by a brief introduction which aids materially in correlating the arithmetic with the subject matter of Elementary Business Training, and serves as a summary of the work learned in the classroom.

The problems in the book have been tried out in class so as to make sure of their usability.

Mastery Arithmetic. Book I. By G. R. Bodley, Chas. Gibson, Ina M. Hayes, and B. M. Watson. D. C. Heath and Co., 1934. VII+335. Price, $0.72. Book II. By the same authors. IX+390. Price, $0.76.

In these two books the first of which is intended for grades 3 and 4 and the second for grades 5 and 6, the authors have tried to present arithmetic so that the pupil will be able with a minimum of suggestion and help from the teacher to pursue his work independently. Throughout each book the pupil is challenged to test his own power, to check his own progress, to correct his own deficiencies, and to grow by his own effort.

Some features of these books are: 1. Careful gradation; 2. Simplification of topics; 3. A quantity of material for teaching and testing more than is considered sufficient for all classroom needs; 4. The use of words as well as of figures for the expression of numbers; 5. Mental tests.

College Algebra. By A. M. Harding and G. W. Mullins. Macmillan, 1936. VII+379. Price, $2.25.

This book contains the material that is usually included in a semester's course in algebra and which is fairly standardized for such a course. Because of the varying needs of different colleges the authors have tried to give the text the greatest flexibility by making most of the chapters independent, so that the omissions of parts of chapters, or even of entire chapters, will not break the continuity of the course.

Special features of this text are: 1. The frequent use of the graphical method; 2. The introduction of the derivative; and 3. An abundance of review material.

An Evaluation of Courses in Education of a State Teachers College by Teachers in Service. By Roscoe George Linder. Bureau of Publications, Teachers College, Columbia University. VII+156. Contributions to Education No. 664. Cloth, $1.85.

In this book two basic criticisms of teachers college curricula are analyzed and applied to the education, psychology, and sociology courses of a state teachers college. From evaluations by graduates and from similar studies, suggestions are offered on the extent of duplication and the choice and treatment of topics for prospective teachers. Administrators and college instructors will find the lists of topics and suggestions useful in the revision of their courses.